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The BGK Model with External Confining Potential: Existence, Long-Time Behaviour and Time-Periodic Maxwellian Equilibria

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Abstract We study global existence and long time behaviour for the inhomogeneous nonlinear BGK model for the Boltzmann equation with an external confining potential. For an initial datum $f_0 \geq 0$ with bounded mass, entropy and total energy we prove existence and strong convergence in L^1 to a Maxwellian equilibrium state, by compactness arguments and multipliers techniques. Of particular interest is the case with an isotropic harmonic potential, in which Boltzmann himself found infinitely many time-periodic Maxwellian steady states. This behaviour is shared with the Boltzmann equation and other kinetic models. For all these systems we study the multistability of the time-periodic Maxwellians and provide necessary conditions on f_0 to identify the equilibrium state, both in L^1 and in Lyapunov sense. Under further assumptions on f, these conditions become also sufficient for the identification of the equilibrium in L^1.

Keywords BGK model · Boltzmann equation · External force · Long time behaviour · Maxwellian steady states

1 Introduction

We consider the BGK Boltzmann equation [2]:

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f = M[f] - f \tag{1.1}$$

with $(t, x, v) \in (0, +\infty) \times \mathbb{R}^N \times \mathbb{R}^N$, where the *local Maxwellian* $M[f]$

$$M[f](t, x, v) = \frac{\rho(t, x)}{(2\pi T(t, x))^{N/2}} \exp\left(-\frac{|v - u(t, x)|^2}{2T(t, x)}\right) \tag{1.2}$$

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is defined in terms of the velocity moments of f through the spatial density ρ, the mean velocity u and the temperature T

$$\begin{pmatrix} \rho \\ \rho u \\ \rho|u|^2 + \rho T N \end{pmatrix}(t,x) = \int_{\mathbb{R}^N} \begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,x,v)dv.$$

This kinetic model describes the time evolution of a system with a large number of particles in a dilute gas. The left hand side is the transport operator with force field $-\nabla_x \Phi$, while the right hand side describes the interactions between particles. The unknown $f(t,x,v) \geq 0$ represents the probability density of particles that, at time t, are at position x with velocity v. The external potential $\Phi = \Phi(x)$ satisfies the assumptions

$$\Phi(x) \geq 0, \Phi \in C^2(\mathbb{R}^N), \quad \exp(-\Phi(x)) \in L^1(\mathbb{R}^N), \tag{1.3}$$

$$|x||\nabla\Phi(x)| \leq c_1(1+\Phi(x)), \qquad |\nabla\Phi(x)|(1+|v|^\sigma) \leq c_2(1+|v|^2+2\Phi(x)), \tag{1.4}$$

for some $\sigma \in (0,1]$ and $c_1, c_2 \in (0,+\infty)$.

And, $\exists R^* > 0$ such that $\forall R \geq R^*$ the energy level set

$$\Gamma_R = \{(x,v) \in \mathbb{R}^{2N} : |v|^2 + 2\Phi(x) = R\} \quad \text{is a regular } C^2\text{-submanifold of } \mathbb{R}^{2N-1}. \tag{1.5}$$

The hypothesis (1.3) assures the existence of the Hamiltonian flow associated to the transport of (1.1) as well as the presence of non trivial steady states with finite mass and energy, therefore Φ is said to confine the particles. In (1.4)–(1.5) there are technical assumptions concerning the growth of Φ at infinity (which for example can be polynomial of any arbitrarily order). The BGK is a model for the Boltzmann equation having a simplified collision term which preserves the qualitative properties of the true collision operator: conservation of mass and total energy, H-theorem and the Euler and Navier-Stokes hydrodynamic limits (see [29, 30], in absence of potential). For this reason the model (1.1), with constant relaxation time, and its variants result useful for physical considerations (cf. [8]). Recent extensions of the original BGK model have been proposed in [1, 6] and [9]. BGK models find also application in other mathematical fields, such as the kinetic formulation of conservation laws (cf. [3, 26]) and the construction of numerical schemes ([4, 27]).

We consider the Cauchy problem of (1.1) with initial datum

$$f(t=0,x,v) = f_0(x,v) \geq 0 \quad \text{a.e.} \quad \text{in } \mathbb{R}^{2N}, \tag{1.6}$$

having bounded mass, energy and entropy:

$$\int_{\mathbb{R}^{2N}} f_0(1 + 2\Phi(x) + |v|^2 + |\log f_0|)dxdv = c_0 < +\infty. \tag{1.7}$$

The interest in the model (1.1) consists in the study of the long time behaviour of the system. Indeed, the introduction of a confining potential Φ has the scope of keeping the gas trapped even as the time goes to infinity. For a general potential Φ obeying the conditions above, the Maxwellian equilibrium state is $m_s(x,v) = \alpha \exp(-|v|^2/(2\theta) - \Phi(x)/\theta) \in L^1(\mathbb{R}^{2N})$, with $\alpha, \theta > 0$. However there are cases (e.g. $\Phi(x) = |x|^2/2$) in which the system admits infinitely many time-dependent Maxwellian equilibria. They have been computed by Boltzmann for the Boltzmann equation (under quadratic potential) and are reported in several books of statistical physics. Here a first question is if the system is stable with respect to them and in which sense. Then it is natural to ask if the system relaxes towards one of them. Finally one would like to identify the limit only using information on the initial datum f_0 and the conserved quantities.

The first aim of this paper is to analyze the existence and the long time behaviour of the solutions under the influence of an external confining potential satisfying (1.3)–(1.5). Global L^1-existence in $\mathbb{R}^N$ for the BGK equation without potential has been investigated in [25] and [24], while [28] dealt with bounded domains. We follow the work of Perthame [25], (used also in other cases, see [6, 30]). We analyze in particular the effect of our Hamiltonian transport in the estimates and their time dependence, since this is very important for the long time limit.

The H-Theorem then shows that the states with constant logarithmic entropy (the *steady states*) are local Maxwellians. By passing to the limit $t \to +\infty$, we verify that the final effect is a relaxation towards a Maxwellian distribution. This has been proved in the case of bounded domains with so-called thermalizing boundary conditions (cf. [10]) and in the case of a linear relaxation model (cf. [7]). Applying the compactness method of the existence part, we get L^1-strong convergence of the time translated sequence $f(t + t_n, x, v)$ to a Maxwellian steady state $m(t, x, v)$, with the same mass as f_0 and bounded energy and entropy (Theorem 4). This result is unconditioned and can also be used to remove the a-priori assumption used in [10].

In Sect. 5, in dependence on the potential, we discuss the regular Maxwellian steady state solutions for the equation, the so-called *global Maxwellians*. Of particular interest is the above-mentioned quadratic case $\Phi(x) = \sum a_j x_j^2 + b \cdot x$, in which Boltzmann found infinite time-dependent Maxwellian solutions. This behaviour is peculiar to the whole-space problem and it is shared with a full class of kinetic equations

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f = C(f) \tag{1.8}$$

with $C(f)$ a collision operator of Boltzmann-type described in (5.1) (since all models have the same steady states). We focus our attention on the isotropic harmonic potential $|x|^2/2$, where the solutions are time periodic. The anisotropic case results easier than the previous one, and we only mention it. The computations for the isotropic potential are reported in [8]—Chap. III, where it is said: "the above result, ..., shows that equilibrium is not necessarily achieved in an harmonic field". Also the problem of the stability of this set has remained open.

The second aim of our work is then to provide some answers for the harmonic field: for the class of kinetic equations (1.8) we determine the multistability for the global Maxwellians in L^1 and in Lyapunov sense (Lemma 3). For the BGK model, as already mentioned, we can say more since we prove convergence to equilibrium in L^1. The compactness method of Sect. 4 is anyway unable to uniquely identify the limit m. Hence, in Sect. 5 we investigate the Maxwellian steady states of (1.8). Our first step is the study of the evolution of some moments of order 1 and 2 of $f(t)$, which result time-periodic. They let us identify a unique Maxwellian $m_\infty(t)$, for which we give a characterization of the basin of attraction (both in L^1 and in Lyaponov sense). This equilibrium state depends on f_0 and minimizes the relative entropy $H[f(t), m(t)]$, among all the Maxwellian solutions m with the same mass as f_0 (Proposition 1). On the other hand, the stationary (in sense of time-independent) equilibrium state m_s minimizes the entropy $H[m(t)]$. We then compare the properties of $m_\infty(t)$ and m_s. In Sect. 6, under some assumptions on f, we finally show that the necessary conditions are also sufficient to prove that $f(t + t_n) \to m_\infty(t)$ in L^1, for the BGK. In the Appendix we collect some computations for the harmonic potential and some details of the existence theorem.

2 Existence

We introduce the main existence theorem for the BGK Boltzmann equation.

Theorem 1 *Let (1.1)–(1.7) define the Cauchy problem for the BGK Boltzmann equation. Then, there exists a nonnegative mild solution f, with $(1+2\Phi(x)+|v|^2)f \in C([0,+\infty); L^1(\mathbb{R}^{2N}))$ and such that*

$$\int_{\mathbb{R}^{2N}} (1+\Phi(x)+|v|^2+|\log f|) f\,dx\,dv \le c(t_0) < +\infty, \quad \forall t \le t_0. \tag{2.1}$$

Moreover, f satisfies the global conservation of mass and total energy, and (ρ, u, T) solve in distributional sense the following "hydrodynamical system"

$$\begin{aligned} &\partial_t \rho + \nabla_x \cdot (\rho u) = 0, \\ &\partial_t(\rho u) + \nabla_x \cdot \Big(\int f v \otimes v\,dv \Big) + \rho \nabla_x \Phi = 0, \\ &\partial_t(\rho |u|^2 + N\rho T) + \nabla_x \cdot \Big(\int f v |v|^2 dv \Big) + 2\nabla_x \Phi \cdot (\rho u) = 0 \end{aligned} \tag{2.2}$$

where the last equation is valid only for potentials with $\sigma = 1$ in (1.4).

If $\Phi = \Phi(|x|)$ is a radial potential and $\int_{\mathbb{R}^{2N}} f_0 |x|^2 dx\,dv < \infty$ holds, then f satisfies the conservation of the angular momentum components $\int_{\mathbb{R}^{2N}} f(x_j v_k - x_k v_j)dx\,dv$, for each $j, k = 1, \dots, N$.

The proof of Theorem 1 is included in Appendix 7.2, and here we only show the main points of it.

Before proving this result, we recall some facts about the transport equation:

$$(\partial_t + v \cdot \nabla_x - \nabla_x \Phi \cdot \nabla_v) f = g - f, \quad f(t=0) = f_0 \tag{2.3}$$

with $g \in L^\infty_{loc}([0,+\infty); L^1(\mathbb{R}^{2N}))$. Under the hypothesis on Φ, the Hamiltonian system

$$\begin{cases} \frac{d}{ds} X(s;t,x,v) = V(s;t,x,v), & X(t;t,x,v) = x, \\ \frac{d}{ds} V(s;t,x,v) = -\nabla_X \Phi(X(s;t,x,v)), & V(t;t,x,v) = v \end{cases}$$

defines a unique classical flow $(X(s),\ V(s))$, which preserves the measure due to the conservation of energy $|V(s;t,x,v)|^2 + 2\Phi(X(s;t,x,v)) = |v|^2 + 2\Phi(x), \forall s < \infty$, and the Jacobian $J(s;t,x,v)$ of the map $(x,v) \mapsto (X(s;t,x,v), V(s;t,x,v))$ is identically one. Therefore, the unique solution $f \in C([0,+\infty); L^1(\mathbb{R}^{2N}))$ of (2.3) is given by

$$\begin{aligned} f(t,x,v) &= e^{-t} f_0(X(0;t,x,v), V(0;t,x,v)) \\ &\quad + \int_0^t e^{s-t} g(s, X(s;t,x,v), V(s;t,x,v))ds, \end{aligned} \tag{2.4}$$

or equivalently, for $t_2 \ge t_1 \ge 0$,

$$f^\#(t_2,x,v) = e^{t_1-t_2} f^\#(t_1,x,v) + \int_{t_1}^{t_2} e^{s-t_2} g^\#(s,x,v)ds, \tag{2.5}$$

where the notation $h^\#(s,x,v) = h(s, X(s;t,x,v), V(s;t,x,v))$ denotes the restriction to the characteristics.

2.1 Moment Estimates and Velocity Averaging

In this section we collect the two main technical results of the existence proof. The first result concerns the boundedness of the v-moment of order $(2+\sigma)$ of f in bounded x-domains and in terms of lower moments. This estimate will be also necessary in Sect. 4.

Lemma 1 *Let $f \in C([0,+\infty); L^1(\mathbb{R}^{2N}))$ solve (2.3) with $f_0 \geq 0$, $g \geq 0$ a.e. such that*

$$\int_{\mathbb{R}^{2N}} (1+|v|^2+\Phi(x))\big(f_0(x,v)+g(t,x,v)\big)dxdv \leq b_0 < +\infty$$

$\forall t \in [t_1,t_2] \subset [0,+\infty)$, where Φ satisfies (1.3)–(1.5) and $b_0 = b_0(t_2,t_1)$ is a constant. Then, for any bounded subset K_x of $\mathbb{R}^N_x$ it holds

$$\int_{t_1}^{t_2}\int_{K_x\times\mathbb{R}^N} |v|^{2+\sigma} f dxdvdt \leq b_1, \tag{2.6}$$

with σ given in (1.4) and $b_1 = b_1(b_0,t_1,t_2,\operatorname{diam}(K_x))$ a finite constant.

Proof Under the hypothesis, (2.3) has the nonnegative solution (2.4) that satisfies the same estimates as g. We multiply (2.3) by the C^1-function

$$\varphi(x,v) = (1+|v|^2)^{\sigma/2}\frac{(x-x_0)\cdot v}{(1+|x-x_0|^2)^{1/2}}, \qquad \text{with } x_0 \in K_x \text{ a fixed point,}$$

and we integrate by parts over $(t_1,t_2)\times\mathbb{R}^N_x\times\mathbb{R}^N_v$ (cf. [23], [17]):

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^{2N}} f\frac{|v|^2(1+|v|^2)^{\sigma/2}}{(1+|x-x_0|^2)^{1/2}}\left(1-\frac{(v\cdot(x-x_0))^2}{(1+|x-x_0|^2)|v|^2}\right)dxdvdt\\
&= \int_{\mathbb{R}^{2N}}(1+|v|^2)^{\sigma/2}\frac{v\cdot(x-x_0)}{(1+|x-x_0|^2)^{1/2}}(f(t_2)-f(t_1))\,dxdv\\
&\quad -\int_{t_1}^{t_2}\int_{\mathbb{R}^{2N}}(1+|v|^2)^{\sigma/2}\frac{v\cdot(x-x_0)}{(1+|x-x_0|^2)^{1/2}}(g-f)dxdvdt\\
&\quad +\int_{t_1}^{t_2}\int_{\mathbb{R}^{2N}} f\frac{(1+|v|^2)^{\sigma/2}\nabla_x\Phi}{(1+|x-x_0|^2)^{1/2}}\cdot\left(\sigma v\frac{(x-x_0)\cdot v}{1+|v|^2}+(x-x_0)\right)dxdvdt\\
&=: I_1+I_2+I_3.
\end{aligned}$$

The left hand side can be bounded from the bottom by

$$\begin{aligned}
&\int_{t_1}^{t_2}\int_{\mathbb{R}^{2N}}\frac{|v|^2(1+|v|^2)^{\sigma/2}}{(1+|x-x_0|^2)^{\frac{3}{2}}}f dxdvdt\\
&\quad\geq \frac{1}{(1+\operatorname{diam}(K_x)^2)^{\frac{3}{2}}}\int_{t_1}^{t_2}\int_{K_x\times\mathbb{R}^N}|v|^{2+\sigma}f dxdvdt.
\end{aligned}$$

We conclude by estimating from above the three integrals at the right hand side

$$|I_1| \leq \int_{\mathbb{R}^{2N}}\frac{(1+|v|^2)^{\sigma/2}|v||x-x_0|}{(1+|x-x_0|^2)^{1/2}}(f(t_1)+f(t_2))\,dxdv$$

$$\le c \int_{\mathbb{R}^{2N}} (1+|v|^2)\,(f(t_1)+f(t_2))\,dx dv \le c_1(b_0,t_2,t_1)$$

since $\sigma \le 1$. Analogously, we get $|I_2| \le c_2(b_0,t_2,t_1)$. Finally, by (1.4),

$$|I_3| \le c \int_{t_1}^{t_2} \int_{\mathbb{R}^{2N}} |\nabla_x \Phi|(1+|v|^\sigma) f\,dx dv dt$$

$$\le c \int_{t_1}^{t_2} \int_{\mathbb{R}^{2N}} (1+2\Phi(x)+|v|^2) f\,dx dv dt \le c_3(b_0,t_2,t_1).$$

The previous computations can be justified after a regularization of the solution f. □

The second result is a L^1-velocity averaging lemma for Hamiltonian systems, that slightly extends Proposition 3 of [18] (see also [15]). The same result holds in case $\mathbb{R}_t$ is replaced by a time interval (t_1,t_2).

Lemma 2 *Let $\mathcal{K} \subset L^1(\mathbb{R}_t \times \mathbb{R}^N_x \times \mathbb{R}^N_v)$ be a bounded and uniformly integrable set and let $\mathcal{S}$ be the corresponding set of solutions F of the following transport equation*

$$(1+\partial_t + v\cdot\nabla_x - \nabla_x\Phi\cdot\nabla_v)F = G,$$

with $G \in \mathcal{K}$ and $\Phi = \Phi(x)$ a given potential satisfying (1.3). Then, for each $\psi \in W^{1,\infty}(\mathbb{R}^N)$ with compact support, the set of velocity averages $\{\int_{\mathbb{R}^N} F(t,x,v)\psi(v)dv : F \in \mathcal{S}\}$ is compact in $L^1_{loc}(\mathbb{R}_t \times \mathbb{R}^N_x)$.

Proof We first apply a localization argument by defining the family $\{f = \xi F : F \in \mathcal{S}\}$ in terms of the function $\xi \in C_0^\infty(\mathbb{R}\times\mathbb{R}^N\times\mathbb{R}^N)$, with $\xi \equiv 1$ on $P \times \mathrm{supp}\psi$, where $\psi \in W^{1,\infty}(\mathbb{R}^N_v)$ is a fixed test function with compact support and $P \subset \mathbb{R}_t \times \mathbb{R}^N_x$. Hence, f solves the linear equation:

$$D(f) = h, \quad D := (1+\partial_t + v\cdot\nabla_x - \nabla_x\Phi\cdot\nabla_v),$$
$$\text{with } h := D(\xi F) = D(\xi)F + \xi G \tag{2.7}$$

and both the families $\{f : F \in \mathcal{S}\}$ and $\{h : F \in \mathcal{S}, G \in \mathcal{K}\}$ have (the same) compact support and are uniformly integrable in $L^1(\mathbb{R}\times\mathbb{R}^{2N})$. One can write the unique solution f in terms of the Hamiltonian flux $X(s), V(s)$ as: $f(t,x,v) = \int_0^\infty e^{-s} h(t-s, X(t-s;t), V(t-s;t))ds (:= R(h))$, where we call $R = D^{-1}$ the resolvent of the equation. It holds (uniformly in $\mathcal{K}$):

$$\|f\|_{L^1(\mathbb{R}\times\mathbb{R}^{2N})} = \|R(h)\|_{L^1(\mathbb{R}\times\mathbb{R}^{2N})} \le \|h\|_{L^1(\mathbb{R}\times\mathbb{R}^{2N})} \le c(|\mathrm{supp}\xi|). \tag{2.8}$$

We denote by $a_\psi = \int_{\mathbb{R}^N} f(t,x,v)\psi(v)dv$ the velocity moment. For a fixed $b > 0$, we perform the decomposition $f = f_b^> + f_b^<$ and $a_\psi = a_{\psi b}^> + a_{\psi b}^<$, for the families of solutions and moments, respectively ($1_{\{\}}$ is the indicator function): $f_b^> = R(h 1_{\{|h|>b\}})$, $f_b^< = R(h 1_{\{|h|\le b\}})$, $a_{\psi b}^>(t,x) = \int_{\mathbb{R}^N} f_b^>(t,x,v)\psi(v)dv$, $a_{\psi b}^<(t,x) = \int_{\mathbb{R}^N} f_b^<(t,x,v)\psi(v)dv$.

From the boundedness and uniform integrability of the set $\{h : F \in \mathcal{S},\ G \in \mathcal{K}\}$ and from (2.8), one obtains for each ψ as $b \to +\infty$, uniformly in $\mathcal{K}$:

$$\|a_{\psi b}^>\|_{L^1(\mathbb{R}\times\mathbb{R}^N)} \le \|\psi\|_{L^\infty(\mathrm{supp}\psi)} \|h 1_{\{|h|>b\}}\|_{L^1(\mathbb{R}\times\mathbb{R}^{2N})} \to 0. \tag{2.9}$$

Then, we define $g_1 = h1_{\{|h|\le b\}} - f_b^<$, $g_2 = \nabla_x \Phi f_b^<$ and rewrite the transport equation (2.7) satisfied by $f_b^<$ in the form

$$(\partial_t + v \cdot \nabla_x) f_b^< = g_1 + \mathrm{div}_v(g_2),$$

where by construction and from (2.8), the terms $f_b^<, g_1, g_2$ are bounded in $L^2(\mathbb{R} \times \mathbb{R}^{2N})$, uniformly in $\mathcal{K}$. The L^2-averaging lemma (cf. Theorem 5 in [13]) implies that $a_{\psi b}^< \in H^{1/4}(\mathbb{R} \times \mathbb{R}^N)$, with uniform bound. Consequently the family $\{a_{\psi b}^< : F \in \mathcal{S}\}$ belongs to a compact set of $L^1(P)$. Combining this with relation (2.9), one finally derives the relatively compactness in $L^1(P)$ for $\{a_\psi : F \in \mathcal{S}\}$, that is for the (truncated) velocity moments $\{1_{\{(t,x)\in P\}} \int F(t,x,v)\psi(v)dv\}$ of F. □

Remark 1 We observe that conditions (1.3)–(1.5) are satisfied by potentials with polynomial growth, i.e. $\Phi(x) = |x|^r$ for $|x| > R^*$ and $r > 0, r \in \mathbb{R}$. In particular, an application of the Young's inequality $ab \le a^p/p + b^q/q$ leads to the second condition in (1.4) with $\sigma = \min(1, 2/r)$ ($\sigma = 1$ for quadratic or subquadratic potentials). Bounded perturbations in this class or lower order potentials are also admissible.

2.2 Further Estimates

Here we show other estimates for the solutions of Theorem 1.

Theorem 2 *Under the setting of Theorem 1, we consider f_0 fulfilling the two additional conditions $(2\Phi(x) + |v|^2)^2 f_0 \in L^1(\mathbb{R}^{2N})$ and $f_0(x,v) \ge g(2\Phi(x) + |v|^2)$ a.e., for some function $g(s) > 0$.*

Then, $\rho(t,x) > 0$, $T(t,x) > 0$ a.e., and for all $t > 0$

$$\int_{\mathbb{R}^{2N}} (2\Phi(x) + |v|^2)^2 f(t) dx dv \le e^{2t/N} \int_{\mathbb{R}^{2N}} (2\Phi(x) + |v|^2)^2 f_0 dx dv.$$

Consequently, the hydrodynamic system (2.2) is satisfied by the velocity moments of f for every potential Φ satisfying conditions (1.3)–(1.4).

More generally, for $a > 1$ and $(2\Phi(x) + |v|^2)^a f_0 \in L^1(\mathbb{R}^{2N})$, there is $c_1 > 0$ s.t.

$$\int_{\mathbb{R}^{2N}} (2\Phi(x) + |v|^2)^a f(t) dx dv \le e^{c_1 t} \int_{\mathbb{R}^{2N}} (2\Phi(x) + |v|^2)^a f_0 dx dv.$$

Proof The estimate on the fourth moment comes from the inequality

$$\begin{aligned}\frac{d}{dt} \int_{\mathbb{R}^{2N}} (2\Phi(x) + |v|^2)^2 f dx dv &= \int_{\mathbb{R}^{2N}} (2\Phi(x) + |v|^2)^2 \big(M[f] - f\big) dx dv \\ &= \int_{\mathbb{R}^{2N}} |v|^4 M[f] dx dv - \int_{\mathbb{R}^{2N}} |v|^4 f dx dv \\ &\le \frac{2}{N} \int_{\mathbb{R}^{2N}} (2\Phi(x) + |v|^2)^2 f dx dv.\end{aligned}$$

and Gronwall's lemma. Indeed we get, using spherical coordinates and Jensen's inequality

$$\int_{\mathbb{R}^{2N}} |v|^4 M[f] dx dv = \int_{\mathbb{R}^N} \rho\big(N^2T^2 + 2NT^2 + |u|^4 + 2N|u|^2 T + 4T|u|^2\big) dx$$

$$
\begin{aligned}
&= \int_{\mathbb{R}^N} \rho(NT+|u|^2)^2 + \frac{2}{N}\rho[(NT+|u|^2)^2 - |u|^4]dx \\
&= \Big(1+\frac{2}{N}\Big)\int_{\mathbb{R}^N} \rho\Big(\int_{\mathbb{R}^N} |v|^2 \frac{f}{\rho} dv\Big)^2 dx - \frac{2}{N}\int_{\mathbb{R}^N} \rho|u|^4 dx \\
&\le \Big(1+\frac{2}{N}\Big)\int_{\mathbb{R}^{2N}} |v|^4 f dv dx - \frac{2}{N}\int_{\mathbb{R}^{2N}} |u|^4 f dv dx,
\end{aligned}
$$

which substituted above proves the result. Note that here and above we have used that $\rho(t) > 0$ and that $T(t) > 0$, for $t > 0$. This follows from the assumption on f_0 and the mild formulation

$$
f^{\#}(t) = e^{-t} f_0 + \int_0^t e^{s-t} M[f]^{\#}(s) ds \ge e^{-t} g(2\Phi(x) + |v|^2) > 0 \quad \forall (t,x,v),
$$

and using that $g(2\Phi(x)+|v|^2) = g^{\#}(2\Phi(x)+|v|^2)$. The bound of the 4th moment can now be used to control all terms in (2.2) (see proof of Theorem 1). For the case $a > 1$ see also [24]—(A.16). □

3 H-Theorem and Stability

The stability properties for the BGK system can be analyzed in terms of the logarithmic entropy introduced in the proof of the existence. Given two functions $f, g \in L^1(\mathbb{R}^{2N})$ we denote by $H[f] = \int_{\mathbb{R}^{2N}} f \log f dx dv$ the *entropy* of f and by

$$
H[f,g] = \int_{\mathbb{R}^{2N}} f \log \frac{f}{g} dx dv
$$

the *relative entropy* of f with respect to g, where $\|f\|_{L^1} = \|g\|_{L^1}$. This last Lyapunov functional is known to be non negative and vanishes only when $f = g$ a.e. We also recall the Csiszár-Kullback's inequality

$$
\|f-g\|^2_{L^1(\mathbb{R}^{2N})} \le 2\|f\|_{L^1(\mathbb{R}^{2N})} H[f,g] \quad \text{with } \|f\|_{L^1} = \|g\|_{L^1}. \tag{3.1}
$$

The main properties of the entropy and the definition of steady state are treated in the so-called H-Theorem.

Theorem 3 (H-Theorem) *For a solution f of Theorem 1 holds the equality*

$$
\partial_t (f \log f) + v \cdot \nabla_x (f \log f) - \nabla_x \Phi \cdot \nabla_v (f \log f) = (M[f] - f)(1 + \log f) \tag{3.2}
$$

in distributional sense. For any $t_2 > t_1 \ge 0$, one obtains

$$
H[f(t_1)] - H[f(t_2)] = \int_{t_1}^{t_2} \int_{\mathbb{R}^{2N}} (f - M[f])(\log f - \log M[f]) dv dx ds \tag{3.3}
$$

$$
= \int_{t_1}^{t_2} \int_{\mathbb{R}^{2N}} (f - M[f]) \log f dv dx ds \ge 0 \tag{3.4}
$$

with equality if and only if $f(t) = M[f](t)$ *a.e. in* $\mathbb{R}^{2N}$, $\forall t \in [t_1, t_2]$. *In this last case we say that the solution* f *is a* steady state *for* (1.1). *Further, there exists a constant* d_0 *independent of time such that*

$$d_0 \geq H[f_0] - H[f(t)] \geq \int_0^t H[f(s), M[f](s)]ds. \tag{3.5}$$

The H-Theorem and (3.1) imply the Lyapunov stability and the L^1-norm stability for the regular steady states (see Lemma 3 for a general proof).

Corollary 1 *For a solution* $f(t)$ *and a steady state* $m(t)$ *of* (1.1), *such that* $\|m(0)\|_{L^1} = \|f_0\|_{L^1}$, $m \in C^1([0, +\infty) \times \mathbb{R}^{2N})$ *and* $|\log m(t, x, v)| \leq c(1 + 2\Phi(x) + |v|^2)$ $(c \in \mathbb{R})$, *it holds*

$$\begin{aligned} &H[f_0, m(0)] - H[f(t), m(t)] = H[f_0] - H[f(t)] \geq 0, \\ &\|f(t) - m(t)\|_{L^1}^2 \leq 2\|f_0\|_{L^1(\mathbb{R}^{2N})} H[f(0), m(0)], \quad t \geq 0. \end{aligned} \tag{3.6}$$

4 Convergence to Equilibrium

In this section we study the long time behaviour of (1.1)–(1.7). By a compactness argument we show that the sequence $f(t + t_n, x, v)$ converges strongly in $C([0, \tau]; L^1(\mathbb{R}^{2N}))$ to a Maxwellian $m(t, x, v)$. This method, due to L. Arkeryd, has been widely used in the literature of kinetic models [5, 10, 14, 16, 20–22]. We proceed as in the work [10] concerning the BGK equation with reverse or specular reflecting boundary conditions, but here we remove the additional assumption on the solution

$$\sup_{t \in [0, \infty[} \int_{x \in \Omega} \int_{v \in \mathbb{R}^N} f(t, x, v)|v|^3 \, dxdv < +\infty,$$

since Lemma 1 gives the expected control for the higher moments.

Theorem 4 *Let f be a solution of the BGK system* (1.1)–(1.7) *in the sense of Theorem* 1. *Then, for every sequence* t_n *going to infinity, there exists an increasing subsequence* t_{n_k} *and a time dependent local Maxwellian* $m(t, x, v)$ *such that* $f_{n_k}(t, x, v) = f(t_{n_k} + t, x, v)$ *converges strongly in* $C([0, \tau]; L^1(\mathbb{R}^{2N}))$ *to* $m(t, x, v)$, *for every* $0 < \tau < +\infty$. *Further,* $m(t, x, v) \in C([0, +\infty); L^1(\mathbb{R}^{2N}))$ *is a nonnegative mild solution of the equation*

$$\partial_t m + v \cdot \nabla_x m - \nabla_x \Phi \cdot \nabla_v m = 0 \tag{4.1}$$

with initial datum $m(0, x, v) = \lim_{n_k \to \infty} f(t_{n_k}, x, v)$ (*in* $L^1(\mathbb{R}^{2N})$). $m(t)$ *has the same mass as* f_0 *and satisfies*

$$\int_{\mathbb{R}^{2N}} (|v|^2 + 2\Phi(x)) m(t, x, v) dxdv \leq \int_{\mathbb{R}^{2N}} (|v|^2 + 2\Phi(x)) f_0(x, v) dxdv, \tag{4.2}$$

$$H[m(t)] \leq \lim_{t_{n_k} \to \infty} H[f_{n_k}(t)] = \lim_{t_{n_k} \to \infty} H[M[f_{n_k}](t)] \quad \textit{for a.e. } t. \tag{4.3}$$

For radial potentials $\Phi = \Phi(|x|)$ *with more than quadratic growth* (*i.e.* $|x|^{2+\beta} \leq c(1 + \Phi(x))$, *with* $\beta > 0$ *and* $|x| > R$), *m has componentwise the same angular momentum as* f_0.

Proof We divide the proof into three steps.

Step 1: Weak convergence in L^1. For $t_n \to +\infty$, up to an increasing subsequence t_{n_k}, we get $f_{n_k} \rightharpoonup m$ weakly in $L^1([0,\tau]\times\mathbb{R}^{2N})$. This follows from the uniform estimate provided by the conservation laws and the H-Theorem:

$$\sup_{t\in[0,+\infty)} \int_{\mathbb{R}^{2N}} f(t)(1+2\Phi(x)+|v|^2+|\log f(t)|) < +\infty. \tag{4.4}$$

On the other hand, due to the conservation laws, an application of Lemma 1 to (1.1) yields $b_1 = (t_2-t_1)b_2$ in the estimate (2.6), with $b_2 = b_2(b_0, \operatorname{diam}(K_x))$ a constant independent of time. This gives

$$\int_0^\tau \int_{K_x}\int_{\mathbb{R}^N} |v|^{2+\sigma} f_{n_k}\, dx dv dt = \int_{t_{n_k}}^{t_{n_k}+\tau} \int_{K_x}\int_{\mathbb{R}^N} |v|^{2+\sigma} f dx dv dt \le \tau b_2$$

uniformly in n_k. Therefore, as in the proof of Theorem 1, we deduce that $M[f_{n_k}] \to M[m]$ strongly in $L^1([0,\tau]\times\mathbb{R}^{2N})$. Moreover, all the properties in (7.7) hold by replacing (f_ϵ, f) with (f_{n_k}, m). It remains to show that m is a Maxwellian. As in [10], the (3.3)–(3.5) imply for $k\to\infty$

$$\int_0^\tau \int_{\mathbb{R}^{2N}} \left(M[f_{n_k}] - f_{n_k}\right)\left(\log M[f_{n_k}] - \log f_{n_k}\right) dx dv dt \to 0. \tag{4.5}$$

Positivity and convexity of the function $F(x,y) = (x-y)(\log x - \log y)$ finally imply $M[m] = m$ a.e.

Step 2: Strong convergence. Using Jensen's inequality in the time variable, Csiszár-Kullback's inequality (3.1) and the H-Theorem we obtain

$$\begin{aligned} &\frac{1}{2\tau\|f_0\|_{L^1}} \|f_{n_k} - M[f_{n_k}]\|^2_{L^1([0,\tau]\times\mathbb{R}^{2N})} \\ &\quad\le \frac{1}{2\|f_0\|_{L^1}} \int_0^\tau \|f_{n_k} - M[f_{n_k}]\|^2_{L^1(\mathbb{R}^{2N})} dt \le \int_0^\tau H[f_{n_k}, M[f_{n_k}]] dt \\ &\quad\le \int_0^\tau \int_{\mathbb{R}^{2N}} \left(M[f_{n_k}] - f_{n_k}\right)\left(\log M[f_{n_k}] - \log f_{n_k}\right) dx dv dt. \end{aligned} \tag{4.6}$$

In this way, (4.5) implies $\|f_{n_k} - M[f_{n_k}]\|_{L^1([0,\tau]\times\mathbb{R}^{2N})} \to 0$ when $k\to\infty$. In the previous step we have shown that $\|M[f_{n_k}] - m\|_{L^1([0,\tau]\times\mathbb{R}^{2N})} \to 0$ when $k\to\infty$. Therefore f_{n_k} converges strongly to m in $L^1([0,\tau]\times\mathbb{R}^{2N})$. Finally, from the mild formulation

$$\begin{aligned} e^{t_2} f(t_{n_k}+t_2, X(t_2), V(t_2)) &= e^{t_1} f(t_{n_k}+t_1, X(t_1), V(t_1)) \\ &\quad + \int_{t_1}^{t_2} e^s M[f](t_{n_k}+s, X(s), V(s)) ds \end{aligned}$$

we can deduce the time-equicontinuity of the sequence $\{f(t_{n_k}+t, X(t), V(t))\}_{n_k}$. This one also converges to $m(t, X(t), V(t))$ strongly in $L^1([0,\tau]\times\mathbb{R}^{2N})$, which implies its strong convergence in $C([0,\tau]; L^1(\mathbb{R}^{2N}))$ from Ascoli-Arzela's Theorem. As consequence, $f_{n_k}(t,x,v) \to m(t,x,v)$ in $C([0,\tau]; L^1(\mathbb{R}^{2N}))$.

Step 3: Properties of m. In Step 1 we have said that (7.7) holds for the sequence (f_{n_k}, m). Hence, it follows the convergence of the mass and the inequality (4.2) (with equality in the integration domains $K_x \times \mathbb{R}^N_v$). Convexity arguments, the equality $H[f, M[f]] = H[f] - H[M[f]]$ and (3.6) lead to (4.3), where lim inf is replaced by a limit since $H[f_{n_k}](t)$ decreases in n_k from the H-Theorem. Finally, for a super-quadratic radial potential, the estimates (4.4) and the pointwise convergence of f_{n_k} to m yield $(x_j v_k - x_k v_j) f_{n_k}(t) \to (x_j v_k - x_k v_j) m(t)$ in $L^1(\mathbb{R}^{2N})$. □

Remark 2 (a) As in [10], one can derive a "preliminary" speed of convergence for the time averages, showing how fast f approaches its local Maxwellian:

$$\frac{1}{t}\int_t^{2t} \|f - M[f]\|^2_{L^1(\mathbb{R}^{2N})}\, ds \le \frac{c}{t}. \tag{4.7}$$

We can write (4.7) as $\|f(t^*) - M[f(t^*)]\|_{L^1(\mathbb{R}^{2N})} \le c/\sqrt{t}$ for some $t^* \in [t, 2t]$. Anyway this result cannot be used to derive an explicit convergence rate to the equilibrium Maxwellian m. This might be achieved with other methods (hypocoercivity, entropy estimates). They require strong regularity and time independent bounds, which have not yet been proved even for the BGK in the torus.

(b) As in [10, Lemma 2], we can also show that the Maxwellian m of Theorem 4 has unbounded support (i.e. $m > 0$ a.e.). Call ρ_m the density of m and $A \subset [0, +\infty) \times \mathbb{R}^N$ a set with positive measure where $\rho_m(t, x) > 0$. Up to a set K of zero measure, one gets $m(t, x, v) = m_0(X(0; t, x, v), V(0; t, x, v)) > 0, \forall (t, x, v) \in A \times \mathbb{R}^N - K$. Since m is a Maxwellian, we get $m > 0$ if and only if $\rho_m > 0$. Hence, we look for a $\tilde{v}$ such that $m(t, x, \tilde{v}) > 0$, because in this way $\rho_m(t, x) > 0$, and consequently $m(t, x, v) > 0$ for a.e. $v \in \mathbb{R}^N$. For any (t, x) with $(t, x, v) \in [0, +\infty) \times \mathbb{R}^{2N} - K$ we have to find $(t_*, x_*, v_*) \in A \times \mathbb{R}^N - K$ and $\tilde{v}$ such that

$$X(0; t, x, \tilde{v}) = X(0; t_*, x_*, v_*) \qquad V(0; t, x, \tilde{v}) = V(0; t_*, x_*, v_*). \tag{4.8}$$

This immediately implies $m(t, x, \tilde{v}) = m_0(X(0; t, x, \tilde{v}), V(0; t, x, \tilde{v})) = m_0(X(0; t_*, x_*, v_*), V(0; t_*, x_*, v_*)) = m(t_*, x_*, v_*) > 0$, since $(t_*, x_*, v_*) \in A \times \mathbb{R}^N - K$ and consequently $\rho_m(t, x) > 0$. Assuming

The external potential is such that the following set has zero measure

$$\Gamma = \left\{(t, x) \in \mathbb{R} \times \mathbb{R}^N - K_1 | \forall (t_*, x_*) \in A, (4.8) \text{ has no solution}\right\} \tag{4.9}$$

we can prove that the sets where either (4.8) fails (i.e. Γ) or where (4.8) holds with $(t_*, x_*, v_*) \in K$, have zero measure. Hence, (4.9) yields $m > 0$.

5 Maxwellian Steady States for the Boltzmann and BGK Equations

In this section we deal with the regular steady states solutions for (1.1), called global Maxwellians, and their connection with the results of the previous section. This is the point in which the behaviour of the BGK in the whole space with confinement qualitatively differs from the BGK in a bounded domain ([10]).

The results shown in this section are applicable to a generic kinetic equation

$$\partial_t f + v \cdot \nabla_x f - \nabla_x \Phi \cdot \nabla_v f = C(f) \quad f(t = 0) = f_0, \tag{5.1}$$

with $C(f)$ a collision operator such that:

1. $1, v, |v|^2$ are collision invariants: $\int_{\mathbb{R}^N} C(f)(1, v, |v|^2)\, dv = 0$; which implies conservation of mass, total energy and, in case $\Phi = \Phi(|x|)$, of angular momentum. Furthermore C must preserve the positivity.
2. The H-theorem holds in this form:

$$H[f](0) - H[f](t) = \int_0^t \int_{\mathbb{R}^{2N}} C(f(s))(-\log f(s)) dx dv ds \geq 0$$

with equality if and only if $C(f) = 0$, and this happens if and only if $f = M[f]$ is a local Maxwellian distribution.

Apart from our model (1.1), this class includes the Boltzmann equation with confining potential. For our purposes we shall simply assume that (5.1) has solutions for which the following considerations make a sense. The results are in particular valid for the constructed solutions of the BGK equation (1.1). As in Sect. 4, the solutions with constant entropy will be denoted *steady states*, while the term *stationary* is always referred to a generic time independent function. According to the H-theorem, at the equilibrium the distribution function f must be a local Maxwellian steady state

$$m(t, x, v) = \rho_m(t, x) \frac{1}{(2\pi T_m(t, x))^{N/2}} \exp\left(\frac{-|v - u_m(t, x)|^2}{2T_m(t, x)}\right), \tag{5.2}$$

solving both $C(m) = 0$ and the linear transport equation

$$\partial_t m + v \cdot \nabla_x m - \nabla_x \Phi \cdot \nabla_v m = 0. \tag{5.3}$$

If we restrict ourselves to classical solutions, then the problem has been solved by Boltzmann himself (for the Boltzmann equation), who considered the case of a more general time dependent forcing term $F(t, x) = -\nabla_x \Phi(x) + x \cdot W(t, m)$, with $W(t, m)$ a special tensor dependent of time and of m itself (cf. [8], Chap. III.10, equation (10.16)). In our case of a time-independent potential, we have $W(t, m) = 0$ and his result reads as follows:

For a quadratic potential $\Phi(x) = \sum_i a_i x_i^2 + b \cdot x$, (5.3) admits an infinite family of time dependent Maxwellian solutions.

We postpone to Sect. 5.2 the classification and the discussion of these time dependent Maxwellians for the harmonic potential. In particular we deal with the isotropic harmonic case, which has time-periodic solutions. Their presence constitutes a problem for the identification of the equilibrium state. Anyway we know that, in terms of Theorem 4, they do not prevent the BGK system (1.1) to converge to a Maxwellian equilibrium state, strongly in $L^1(\mathbb{R}^{2N})$. Further, we can also show that they are Lyapunov- and L^1-stable with respect to the solutions of the BGK and more in general of (5.1) (see Corollary 1 and Lemma 3).

If we consider instead a generic stationary potential Φ, then the only regular Maxwellian solving (5.3) is the so-called *barometric distribution*

$$m_s(x, v) = \alpha \exp\left(-\frac{\Phi(x)}{\theta} - \frac{|v|^2}{2\theta}\right), \quad \text{with } \alpha, \theta \in (0, +\infty). \tag{5.4}$$

If Φ has radial symmetry, then the stationary state is

$$m_s(x, v) = \alpha \exp\left(-\frac{\Phi(x)}{\theta} + \sum_{j,k=1}^{N} w_{jk} x_j v_k - \frac{|v|^2}{2\theta}\right), \quad \text{with } \alpha, \theta \in (0, +\infty), \tag{5.5}$$

where $\{w_{jk}\}$ is an antisymmetric $N \times N$ matrix.

On the other hand, if a stationary potential fulfils a certain condition involving Φ and the parameters of the Maxwellian $m(t,x,v)$ (cf. [8], Chap. III.10, equation (10.20) stated in $\mathbb{R}^3$, but valid also in $\mathbb{R}^N$), then one gets additional time-dependent steady states. This condition can be reformulated as a system of linear equations involving the derivatives of Φ of order 3 and 4 (cf. equation (10.21) in [8]). A complete classification of such Φ is missing. However, according to this computation, a polynomial potential satisfies this condition only in the quadratic case.

In a bounded domain or with linearized models the situation is completely different. For the BGK and Boltzmann equations with thermalizing boundary conditions (specular reflection, reverse reflection, periodic box) and without external potential, Desvillettes [10] found only stationary equilibrium states with constant density $m(x,v) = r_0 \exp(-\nu|v|^2)$ or $m(x,v) = r_0 \exp(-\nu|v|^2 - 2(\lambda_0 z \times x) \cdot v)$ for surfaces of revolutions. Moreover, the linear relaxation-time model in $\mathbb{R}^{2N}$ of [7] with confining potential admits only the classical stationary state (5.4).

The rest of the section is devoted to the classification of the steady states in terms of their moments and their entropy, always with the intention of finding conditions on the initial datum f_0 to permit an identification of the equilibrium state to which $f(t)$ converges for $t \to \infty$.

In the following we shall call $\mathcal{G}(\Phi)$ the family

$$\mathcal{G}(\Phi) = \{m : m(t,x,v) > 0 \text{ is a local Maxwellian (5.2), solution of (5.3),}$$
$$\text{such that } \|m\|_{L^1(\mathbb{R}^{2N})} = \|f_0\|_{L^1(\mathbb{R}^{2N})}, \text{ and with finite total energy}\}. \tag{5.6}$$

For general potentials one gets $\mathcal{G}(\Phi) = \{m_s, \text{ with the same mass as } f_0\}$. In the case $\mathcal{G} := \mathcal{G}(|x|^2/2)$ there are other time dependent elements $m(t)$.

5.1 Stability of the Solutions

We then give a general stability statement, valid for all potentials. The part involving the $m(t)$ Maxwellians is clearly related to the particular potentials admitting them. As first consequence, we get Lyapunov-stability and L^1-norm stability for a solution f of (5.1). Furthermore, if we fix f and let m vary in $\mathcal{G}(\Phi)$ we can study the relation between the several relative entropy functionals $H[f,m](t)$. All of them do not increase in time, which means that the solution $f(t)$ approaches all of them (or better, does not depart from them) during the time evolution. In the next section we shall see that there is only one candidate for the equilibrium and that it depends on f_0.

Lemma 3 *Let $f(t)$ be a solution of (5.1) with initial datum f_0.*

Then, the following statements hold:

(a) *If $m \in \mathcal{G}(\Phi)$ then the following quantities are equal:*

$$\frac{d}{dt} H[f(t), m(t)] = \frac{d}{dt} H[f(t), m_s] = \frac{d}{dt} H[f(t)] = \int_{\mathbb{R}^{2N}} C(f) \log(f) dx dv \le 0. \tag{5.7}$$

In particular, $\forall m_1, m_2 \in \mathcal{G}(\Phi)$, $\forall t \ge 0$

(i) $H[m_1(t)] = H[m_1(0)]$, $H[m_1(t), m_2(t)] = H[m_1(0), m_2(0)]$ *and*

$$H[f(t), m_1(t)] - H[f(t), m_2(t)] = H[f_0, m_1(0)] - H[f_0, m_2(0)].$$

(ii) *If* $H[f_0, m_1(0)] \le H[f_0, m_2(0)]$, *then this relation holds for every time*:

$$H[f(t), m_1(t)] \le H[f(t), m_2(t)], \quad \forall t \ge 0.$$

(b) *The family* $\mathcal{G}(\Phi)$ *is stable in terms of the* L^1*-norm*:

$$\|f(t) - m(t)\|_{L^1}^2 \le 2\|f_0\|_{L^1} H[f(0), m(0)], \quad t > 0, \ m \in \mathcal{G}(\Phi).$$

Proof (a) Using the expression of the relative entropy, $H[f(t), m(t)] = \int_{\mathbb{R}^{2N}} f \log f \, dx dv - \int_{\mathbb{R}^{2N}} f \log m dx dv$, and the H-theorem for (5.1) one gets

$$\frac{d}{dt} H[f(t), m(t)] = \int_{\mathbb{R}^{2N}} C(f) \log f dx dv - \frac{d}{dt} \int_{\mathbb{R}^{2N}} f \log m dx dv.$$

Furthermore $\frac{d}{dt} \int f \log m dx dv = 0$, because m solves (5.3) and (for some a, b, c)

$$\int_{\mathbb{R}^{2N}} C(f) \log m dx dv = \int_{\mathbb{R}^{2N}} C(f) \left(a(t,x) + b(t,x) \cdot v + c(t)|v|^2\right) dv dx = 0,$$

since (5.1) has $(1, v, |v|^2)$ as collisional invariants. This completes the proof of (5.7).

For the BGK equation (1.1) the result has been introduced in Corollary 1 and we prove it rigorously (see relation (3.6)): from the regularity of m we can choose $\eta(t,x,v) = \log(m)\alpha(t)\beta_R(|v|^2 + 2\Phi(x))$ as test function in (7.8) and note (from the computations above) that $(\partial_t + v \cdot \nabla_x - \nabla_x \Phi \cdot \nabla_v) \log m = 0$ and $(\partial_t + v \cdot \nabla_x - \nabla_x \Phi \cdot \nabla_v)\eta = \log(m)\beta_R(|v|^2 + 2\Phi(x))\partial_t \alpha$. When $R \to +\infty$, then the r.h.s. of (7.8) becomes $\int_0^{+\infty} \alpha(t) \int_{\mathbb{R}^{2N}} \log m(t)(M[f] - f)(t) dx dv dt$, which is finite by the hypothesis on m and vanishes since $M[f]$ and f have the same velocity moments.

(b) The statement is a consequence of (5.7) and the Csiszár-Kullback's inequality (3.1). □

5.2 Isotropic Harmonic Potential

We now focus our attention on the isotropic harmonic potential $\Phi(x) = \frac{|x|^2}{2}$, with $x \in \mathbb{R}^N$, for which the already mentioned case of multi-stability occurs for the kinetic equation (5.1). We start by introducing the solutions computed by Boltzmann, as reported in [8]—Chap. III.10 for the three dimensional case, easily extendable to $\mathbb{R}^N$. We write $\log m$ as a polynomial in v

$$\log m = a(t,x) + b(t,x) \cdot v + c(t,x)|v|^2, \quad a(t,x), c(t,x) \in \mathbb{R}, b(t,x) \in \mathbb{R}^N$$

and insert it in (5.3) to obtain

$$c(t) = c_0 + c_1 \cos(2t) + c_2 \sin(2t),$$

$$b_j(t,x) = 2(c_1 \sin(2t) - c_2 \cos(2t))x_j + (c_3 \cos t + c_4 \sin t) + \sum_{h=1}^{N} w_{jh} x_h,$$

$$a(t,x) = |x|^2(c_0 - c_1 \cos(2t) - c_2 \sin(2t)) + (\mathbb{I} \cdot x)(c_3 \sin t - c_4 \cos t) + c_5,$$

where $w = \{w_{ij}\}$ is an antisymmetric $N \times N$ matrix in Cartesian coordinates, $\mathbb{I} = (1, \dots, 1)^T$ is the unit vector of $\mathbb{R}^N$ and $c_0, c_1, \dots, c_5 \in \mathbb{R}$ are integration constants. Taking into account that $\log m$ can be decomposed as follows (cf. (5.2))

$$a(t,x) + b(t,x) \cdot v + c(t)|v|^2 = \log(\rho_m) - \frac{|u_m - v|^2}{2T_m} - \frac{N}{2}\log(2\pi T_m),$$

we obtain the following expression for the hydrodynamical quantities in (5.2)

$$\begin{aligned}
T_m(t) &= -\frac{1}{2c(t)},\\
u_m(t,x) &= -\frac{1}{2c(t)} b(t,x),\\
\rho_m(t,x) &= \exp\left(a(t,x) + \frac{N}{2}\log\left(\frac{-\pi}{c(t)}\right) - \frac{|b(t,x)|^2}{4c(t)}\right).
\end{aligned}$$

For $t = 0$ we can then write

$$\begin{aligned}
\log m(0,x,v) = (c_0 - c_1)|x|^2 - c_4(\mathbb{I}\cdot x) + c_5 - 2c_2(v\cdot x) + c_3(\mathbb{I}\cdot v)\\
+ \sum_{j,h=1}^{N} v_j w_{jh} x_h + (c_0 + c_1)|v|^2.
\end{aligned} \tag{5.8}$$

Sometimes we shall use the upper indexes $c_0^m, \dots, c_5^m$ to distinguish the coefficients related to m from the ones of another Maxwellian.

We recall that $m(t)$ can be written as the evolution of the initial datum $m(0)$ along the characteristics:

$$\begin{aligned}
&m(t,x,v) = m(0, X(-t), V(-t)),\\
&\quad \text{with } X(t) = x\cos(t) + v\sin(t),\ V(t) = -x\sin(t) + v\cos(t).
\end{aligned} \tag{5.9}$$

In the huge family of Maxwellian solutions computed by Boltzmann we are interested only in the subfamily $\mathcal{G}$. The associated integrability conditions lead to some constraints on the $6 + N(N-1)/2$ coefficients $c_0, \dots, c_5$ (for example $c_0 < 0$). These constraints are studied in the Appendix for a particular subfamily of $\mathcal{G}$ and will be employed later.

In the following we introduce some definitions to shorten the notations.

Definition 1 Let $f(t,x,v)$ be a function in $L^1((1+|x|^2+|v|^2)dxdv)$ and $(\rho, u, T)(t,x,v)$ be the corresponding density, bulk velocity and temperature, then we define

$$\begin{aligned}
I_x(t) &:= \int_{\mathbb{R}^{2N}} f(t,x,v)x\,dv\,dx = \int_{\mathbb{R}^N} \rho(t,x)x\,dx,\\
I_v(t) &:= \int_{\mathbb{R}^{2N}} f(t,x,v)v\,dv\,dx = \int_{\mathbb{R}^N} \rho(t,x)u(t,x)\,dx,\\
L(t) &:= \int_{\mathbb{R}^{2N}} f(t,x,v)(v\cdot x)\,dv\,dx = \int_{\mathbb{R}^N} \rho(t,x)(u(t,x)\cdot x)\,dx,\\
E_{pot}(t) &:= \frac{1}{2}\int_{\mathbb{R}^{2N}} f(t,x,v)|x|^2\,dv\,dx = \frac{1}{2}\int_{\mathbb{R}^N} \rho(t,x)|x|^2\,dx,
\end{aligned}$$

$$E_{kin}(t) := \frac{1}{2}\int_{\mathbb{R}^{2N}} f(t,x,v)|v|^2 dvdx = \frac{1}{2}\int_{\mathbb{R}^N} \rho(t,x)(|u(t,x)|^2 + NT(t,x))dx,$$

$$K_{jh}(t) := \int_{\mathbb{R}^{2N}} f(t,x,v)(v_j x_h - v_h x_j) dvdx \quad \forall j, h = 1, \dots, N,$$

$$D(t) := \int_{\mathbb{R}^{2N}} f(t,x,v) dvdx = \int_{\mathbb{R}^N} \rho(t,x)dx,$$

and let $E_{tot}(t) := E_{kin}(t) + E_{pot}(t)$ be the total energy of $f(t)$ and

$$J_x(t) := \sum_{j=1}^{N} I_x(t)_j = \int_{\mathbb{R}^N} \rho(t,x)(x \cdot \mathbb{I})dx,$$

$$J_v(t) := \sum_{j=1}^{N} I_v(t)_j = \int_{\mathbb{R}^N} \rho(t,x)(u(t,x) \cdot \mathbb{I})dx,$$

be the sum of the components of the vectors I_x and I_v. Sometimes we will use the subscript $_f$ to distinguish the quantities related to f from the ones associated to another function. Finally, we recall the notation $\mathcal{G} := \mathcal{G}(|x|^2/2)$, defined in (5.6).

We now use the previous notations to shortly represent a quantity used in the following sections.

For each $m_1, m \in \mathcal{G}$, we write

$$\begin{aligned}
&\int_{\mathbb{R}^{2N}} m_1(0) \log m(0) dxdv \\
&= 2(c_0^m - c_1^m) E_{pot,m_1}(0) - c_4^m J_{x,m_1}(0) + c_5^m D_{m_1}(0) \\
&\quad - 2c_2^m L_{m_1}(0) + c_3^m J_{v,m_1}(0) + 2(c_0^m + c_1^m) E_{kin,m_1}(0) \\
&\quad + \sum_{1 \le j < h \le N} w_{jh}^m K_{jh,m_1}(0).
\end{aligned} \tag{5.10}$$

5.3 Oscillation of the Moments

We recall that mass, total energy and angular momentum are conserved by the solutions of (5.1). The time evolution of the other moments in Definition 1 can be computed explicitly by solving a system of ODEs, as shown in the following lemma.

Lemma 4 *Let f be a solution of* (5.1) *with $\Phi(x) = |x|^2/2$. Then the following relations hold*:

$$I_x(t) = I_x(0)\cos t + I_v(0)\sin t,$$

$$I_v(t) = I_v(0)\cos t - I_x(0)\sin t,$$

$$L(t) = L(0)\cos(2t) + \frac{L_t(0)}{2}\sin(2t), \qquad L_t(0) = 2E_{tot}(0) - 4E_{pot}(0),$$

$$E_{pot}(t) = E_{pot}(0) + \int_0^t L(\sigma)d\sigma$$

$$= E_{pot}(0) + \frac{L(0)}{2}\sin(2t) - \frac{L_t(0)}{4}\cos(2t) + \frac{L_t(0)}{4},$$

where $L_t = dL(t)/dt$. *Consequently,*

$$J_x(t) = J_x(0)\cos t + J_v(0)\sin t, \quad J_v(t) = J_v(0)\cos t - J_x(0)\sin t.$$

Proof Remember that $\int \big(x, v, (x\cdot v), |x|^2, |v|^2\big) C(f)\,dx\,dv = 0$. The first two relations follow from a multiplication of (5.1) by x and v respectively, and then from an integration in $\mathbb{R}^{2N}$ and a double derivation:

$$\frac{d}{dt}I_x(t) = I_v(t), \qquad \frac{d^2}{dt^2}I_x(t) = -I_x(t).$$

The last two expressions are obtained in an analogous manner (with multipliers $(x\cdot v)$ and $|x|^2$), using that $E_{tot}(t) = E_{kin}(t) + E_{pot}(t) = E_{tot}(0)$:

$$\frac{d}{dt}L(t) = 2E_{kin}(t) - 2E_{pot}(t) = 2E_{tot}(t) - 4E_{pot}(t),$$
$$\frac{d^2}{dt^2}L(t) = -4\frac{d}{dt}E_{pot}(t) = -4L(t). \qquad \square$$

As consequence of this lemma, we observe that for an isotropic potential a solution $f(t)$ of (5.1) has moments of 1st and 2nd order (for example kinetic and potential energy) that oscillate with the same periodicity as the Maxwellian steady states of Sect. 5.2. And we recall that a solution f is not in general time-periodic.

Since each Maxwellian $m \in \mathcal{G}$ is a particular solution of (5.1) with initial datum $m(0)$, we can apply the previous lemma to m obtaining the evolution of the related moments:

$$I_{x,m}(t) = I_{x,m}(0)\cos t + I_{v,m}(0)\sin t,$$
$$I_{v,m}(t) = I_{v,m}(0)\cos t - I_{x,m}(0)\sin t,$$
$$L_m(t) = L_m(0)\cos(2t) + \frac{L_{t,m}(0)}{2}\sin(2t),\ L_{t,m}(0) = 2E_{tot,m}(0) - 4E_{pot,m}(0),$$
$$E_{pot,m}(t) = E_{pot,m}(0) + \frac{L_m(0)}{2}\sin(2t) - \frac{L_{t,m}(0)}{4}\cos(2t) + \frac{L_{t,m}(0)}{4}.$$

We then compare the evolution of the moments of f and m.

If we consider f_0 and $m(0)$ such that

$$\begin{cases} D_m(0) = D_f(0), \\ J_{x,m}(0) = J_{x,f}(0), \\ J_{v,m}(0) = J_{v,f}(0), \\ L_m(t) = L_f(0), \\ E_{pot,m}(0) = E_{pot,f}(0), \\ E_{kin,m}(0) = E_{kin,f}(0), \\ K_{jh,m}(0) = K_{jh,f}(0) \quad \forall j,h = 1,\dots,N, \end{cases} \tag{5.11}$$

then we get the following consequence.

Definition 2 Let f_0 be the initial datum for (5.1). We call $m_\infty(t,x,v)$ the unique Maxwellian in $\mathcal{G}$ solving (5.11).

Lemma 5 *For every initial datum f_0 of* (5.1), *the system* (5.11) *identifies a unique Maxwellian $m \in \mathcal{G}$, i.e. m_∞.*

Furthermore, (5.11) *holds for all $t > 0$*:

$$D_{m_\infty}(t) = D_f(t), \qquad J_{x,m_\infty}(t) = J_{x,f}(t), \qquad J_{v,m_\infty}(t) = J_{v,f}(t),$$
$$L_{m_\infty}(t) = L_f(t), \qquad E_{pot,m_\infty}(t) = E_{pot,f}(t), \qquad E_{kin,m_\infty}(t) = E_{kin,f}(t),$$
$$K_{jh,m_\infty}(t) = K_{jh,f}(t), \quad \forall j, h.$$

Proof Equation (5.11) provides $6 + N(N-1)/2$ independent conditions to fix the parameters $c_0, ..c_5, w_{jh}$ of $m \in \mathcal{G}$. Moreover, such conditions and Lemma 4 imply that the above mentioned moments of f and m_∞ coincide for each $t \geq 0$. □

We first observe that the Maxwellian m_∞ identified by f_0 is in general time-dependent. The case in which m_∞ coincides with a stationary Maxwellian m_s will be investigated in Sect. 5.5 and we will see that in such occurrence the moments of f in Definition 1 are constant in time.

In the previous lemma we let f_0 fix and we looked for m fulfilling (5.11).

On the other hand, if $m \in \mathcal{G}$ is given and we let f_0 vary, then we have found the *basin of attraction* for m, which is defined as the set of initial data f_0 for which a solution of (5.1) converges to m as time tends to infinity. We are interested both in L^1-and relative entropy convergence. Lemma 5 gives the preliminary information that f and m have for each time the same moments (of Definition 1).

According to this observation, the basin of attraction of m is the set

$$\mathcal{BA}(m) := \{f_0 : f_0 \text{ is initial datum for (5.1) and satisfies (5.11)}\}. \tag{5.12}$$

As confirmation of this guess, in the next subsection we show that (5.11) implies:

(i) first, the entropy splitting $H[f(t), m(t)] = H[f(t)] - H[m(t)]$ (see Lemma 6),
(ii) and second, a necessary condition on f_0 to have $H[f(t), m(t)] \to 0$ as $t \to +\infty$ (see Proposition 1).

In Sect. 6 we shall show how these necessary conditions become sufficient for the BGK equation.

5.4 Relative Entropy and Necessary Conditions for Its Convergence

In this section we study the properties of the family $\mathcal{G}$ with the help of the entropy and the relative entropy functionals. We recall that Lyapunov functionals, such as the relative entropy, have been successfully employed to investigate the asymptotic stability for the Boltzmann equation in bounded domains (cf. [11]) once the equilibrium state is known and unique. We do not know any application of this method in case of multistable systems. Our use of this tool is mainly aimed at identifying the equilibrium state. We propose to compare the time evolution of the solution f (of (5.1)) with respect to the set of relative entropies $\{H[\cdot, m] := \int (\cdot \log(\cdot/m)) dx dv | m \in \mathcal{G}\}$, i.e. we study the quantities $H[f(t), m(t)]$. In this way in Proposition 1 the knowledge of the initial datum f_0 lets us identify the element $m_* \in \mathcal{G}$ for which

$$H[f(t), m_*(t)] = \inf_{m \in \mathcal{G}} \{H[f(t), m(t)]\} \quad \forall t \geq 0. \tag{5.13}$$

And we will see that $m_* = m_\infty$, as defined in Definition 2. On the other hand, by fixing m_*, we get information on the initial datum f_0 for which the relative entropy is minimal.

In the following we introduce two preliminary important properties of m_∞ related to the splitting of the relative entropy.

Lemma 6 *Let $f(t)$ be a solution of (5.1) with initial datum f_0, and m_∞ be the unique Maxwellian in $\mathcal{G}$ satisfying the condition (5.11). Then, for all $t \geq 0$*

$$H[f(t), m_\infty(t)] = H[f(t)] - H[m_\infty(t)]. \tag{5.14}$$

Moreover, for each $m \in \mathcal{G}$, $m \neq m_\infty$, we have

$$H[f(t), m_\infty(t)] + H[m_\infty(t), m(t)] = H[f(t), m(t)]. \tag{5.15}$$

Proof The relation (5.14) is due to the fact that, under the assumption (5.11) for m_∞,

$$H[m_\infty(0)] = \int_{\mathbb{R}^{2N}} f_0(x, v) \log m_\infty(0, x, v) dx dv \tag{5.16}$$

which holds for all times, as shown in Lemma 3. To prove (5.16) we write $\log(m_\infty(0, x, v))$ as in (5.8), for the appropriate constants $c_0, c_1, \ldots, w_{jh}$ corresponding to m_∞. In this way,

$$\begin{aligned} H[m_\infty(0)] &= 2(c_0 - c_1) E_{pot,m_\infty}(0) - c_4 J_{x,m_\infty}(0) + c_5 D_{m_\infty}(0) - 2c_2 L_{m_\infty}(0) \\ &\quad + c_3 J_{v,m_\infty}(0) + 2(c_0 + c_1) E_{kin,m_\infty}(0) + \sum_{1 \leq j < h \leq N} w_{jh} K_{jh,m_\infty}(0). \end{aligned}$$

Using (5.11) for $m_\infty(0)$, we can write

$$\begin{aligned} H[m_\infty(0)] &= 2(c_0 - c_1) E_{pot,f}(0) - c_4 J_{x,f}(0) + c_5 D_f(0) - 2c_2 L_f(0) \\ &\quad + c_3 J_{v,f}(0) + 2(c_0 + c_1) E_{kin,f}(0) + \sum_{1 \leq j < h \leq N} w_{jh} K_{jh,f}(0), \end{aligned}$$

and therefore we obtain (5.16).

In the following, we prove (5.15) for $t = 0$. We first use (5.14) to rewrite the left hand site as

$$H[f_0, m_\infty(0)] + H[m_\infty(0), m(0)] = H[f_0] - \int_{\mathbb{R}^{2N}} m_\infty(0) \log m(0) dx dv.$$

Then, we use (5.10) for $m_1 = m_\infty$ and, since $m_\infty(0)$ satisfies (5.11), we obtain

$$\begin{aligned} &\int_{\mathbb{R}^{2N}} m_\infty(0) \log m(0) dx dv \\ &\quad = 2(c_0^m - c_1^m) E_{pot,f}(0) - c_4^m J_{x,f}(0) + c_5^m D_f(0) \\ &\qquad - 2c_2^m L_f(0) + c_3^m J_{v,f}(0) + 2(c_0^m + c_1^m) E_{kin,f}(0) + \sum_{1 \leq j < h \leq N} w_{jh}^m K_{jh,f}(0) \\ &\quad = \int_{\mathbb{R}^{2N}} f_0 \log m(0) dx dv. \end{aligned}$$

Hence, $H[f_0] - \int_{\mathbb{R}^{2N}} m_\infty(0) \log m(0) dx dv = H[f_0, m(0)]$, which coincides with the right hand side of (5.15) at time $t = 0$. The same result holds for each time $t > 0$ using Lemma 3a. Indeed, one gets $H[m_\infty(0), m(0)] = H[m_\infty(t), m(t)]$ and $H[f(t), m(t)] - H[f(t), m_\infty(t)] = H[f_0, m(0)] - H[f_0, m_\infty(0)]$. □

Since all states m in $\mathcal{G}$ are stable we must impose restrictions on the initial data in order to expect entropy convergence to a particular solution m_* in $\mathcal{G}$. It seems natural to consider an initial condition f_0 of (5.1) "closer" to m_* than to any other m in $\mathcal{G}$. We express this closeness to m_* by the following constraint on f_0

$$H[f_0, m_*(0)] < H[f_0, m(0)], \quad \forall m \in \mathcal{G} - \{m_*\}. \tag{5.17}$$

Due to Lemma 3 this property holds for all times $t > 0$. This means that if initially the solution is closer to m_* than to any other state m, then this continues to hold during the time evolution.

In the following statement, we notice that (5.17) is a necessary condition for the H-convergence to m_*. Moreover, the condition (5.17) becomes more explicit, since it is shown to be equivalent to $6 + N(N-1)/2$ constraints on the moments of $m_*(0)$ and f_0.

Proposition 1 *Let f_0 be an initial datum for* (5.1), *f be a corresponding solution and $m_* \in \mathcal{G}$ be a Maxwellian satisfying* (5.17). *Then*:

(a) *The relation* (5.17) *is a necessary condition for the Lyapunov convergence of $f(t)$ to $m_*(t)$; i.e. if $\lim_{t\to+\infty} H[f(t), m_*(t)] = 0$ then condition* (5.17) *is satisfied.*
(b) *For each f_0 given, the relation* (5.17) *holds if and only if $m_*(0)$ fulfils the system* (5.11). *This means that m_* coincides with m_∞ of Definition* 2.
(c) *For each $m_* \in \mathcal{G}$ given, the relation* (5.17) *holds if and only if f_0 belongs to the set $\mathcal{BA}(m_*)$ defined in* (5.12).

Proof (a) This can be proved by contradiction. Indeed, if we suppose that $\exists m \in \mathcal{G} - \{m_*\}$ with $H[f_0, m(0)] \le H[f_0, m_*(0)]$, then Czisar-Kullback's inequality (3.1) implies: $\|f(t) - m_*(t)\|^2_{L^1} \le 2\|f_0\|_{L^1} H[f(t), m_*(t)]$ and $\|f(t) - m(t)\|^2_{L^1} \le 2\|f_0\|_{L^1} H[f(t), m(t)]$. Therefore, if we assume that $H[f(t), m_*(t)] \to 0$ for $t \to \infty$, this implies $H[f(t), m(t)] \to 0$ for $t \to \infty$. Consequently, f converges in the L^1-norm both to m_* and to m, which gives a contradiction.

(b) It is a consequence of Lemma 6. We suppose that $m_*(0)$ fulfils the system (5.11), then, using the cited lemma, we obtain the following relation between the relative entropies

$$H[f(t), m_*(t)] + H[m_*(t), m(t)] = H[f(t), m(t)] \quad \forall t \ge 0.$$

Therefore, the relation (5.17) has been demonstrated, because $H[m_*(0), m(0)] > 0$ (since m_* and m are different functions). To finish the proof we assume that the relation (5.17) holds and we reason by contradiction: we suppose that $m_*(0)$ does not satisfy the system (5.11). This means that there exists another Maxwellian m_∞ in $\mathcal{G} - \{m_*\}$ fulfilling the system (5.11). Applying Lemma 6 to m_∞ and m_*:

$$H[f(t), m_\infty(t)] + H[m_\infty(t), m_*(t)] = H[f(t), m_*(t)] \quad \forall t \ge 0,$$

we get a contradiction since we find $H[f_0, m_\infty(0)] < H[f_0, m_*(0)]$.

(c) If $f_0 \in \mathcal{BA}(m_*)$, then the relation (5.17) follows from (5.15) with $m_\infty = m_*$. On the other hand, if (5.17) holds and $f_0 \notin \mathcal{BA}(m_*)$, then one could find some $m_1 \in \mathcal{G}$ such that $f_0 \in \mathcal{BA}(m_1)$. Hence, from Lemma 6 one obtains

$$H[f(t), m_1(t)] + H[m_1(t), m_*(t)] = H[f(t), m_*(t)] \quad \forall t \geq 0,$$

leading to a contradiction. □

5.5 The Stationary Maxwellian Equilibrium

The necessary condition (5.17) for the convergence of $f(t)$ to a Maxwellian steady state (in relative entropy) identifies a Maxwellian $m_\infty \in \mathcal{G}$ having the same mass, energy and angular momentum as f_0. These are the conserved quantities of (5.1).

In the following lemma we note that there exists another element in $\mathcal{G}$ for which the splitting (5.14) is possible. We introduce its definition.

Definition 3 Let f_0 be the initial datum for (5.1). We call

$$m_s(x, v) = \exp\left(c_0(|x|^2 + |v|^2) + \sum_{h=1..N} w_{jh} x_h v_j + c_5 \right),$$

(with fixed coefficients) the unique stationary Maxwellian in $\mathcal{G}$ having the same mass, energy and angular momentum as f_0.

This Maxwellian m_s has lower entropy than m_∞, while m_∞ is the element minimizing the relative entropy functional.

Lemma 7 *Assume $\Phi(x) = |x|^2/2$. Let f_0 be the initial datum and f be a solution for* (5.1) *and let m_s be as in Definition* 3, *then*

$$H[f(t), m_s] = H[f(t)] - H[m_s], \tag{5.18}$$

$$H[m_s] \leq H[m_\infty(t)] \leq H[f(t)], \tag{5.19}$$

$$H[f(t), m_\infty(t)] \leq H[f(t), m_s], \tag{5.20}$$

where m_∞ has been defined in Definition 2.

Moreover, if f_0 is such that $m_s \neq m_\infty$, then the inequality (5.20) *and the first inequality in* (5.19) *are strict.*

Proof Concerning the uniqueness of m_s, the conservation laws let one fix the free parameters c_0, w_{jh}, c_5 in m_s and determine the stationary equilibrium state of (5.3) uniquely. The (5.18) follows from $\int_{\mathbb{R}^{2N}} f(t) \log m_s dx dv = H[m_s]$, since $\log m_s$ is linear combination of collision invariants, and from the conservation laws. The positivity of the relative entropy implies both $H[f(t)] \geq H[m_\infty(t)]$ and $H[f(t)] \geq H[m_s]$. But (5.18) holds for all solutions $f(t)$ having the same mass, energy and angular momentum as f_0, hence also for $m_\infty(t)$. This implies $H[m_\infty(t)] \geq H[m_s]$, completing the proof of (5.19). By a difference with $H[f(t)]$ and from the equalities (5.14) and (5.18), one directly gets (5.20) (which is also a direct consequence of Proposition 1(b). The strict inequalities are a consequence of two facts: the relative entropy vanishes if and only if both functions coincide, and for both functions m_s

and $m_\infty(t)$ the splitting for the relative entropy holds. Therefore if $H[m_s] = H[m_\infty(t)]$, then $H[m_\infty(t), m_s] = 0$ and $m_s = m_\infty(t)$. In the other case, if $H[f(t), m_\infty(t)] = H[f(t), m_s]$, then $H[m_s] = H[m_\infty(t)]$ implying $m_s = m_\infty(t)$. □

To summarize: the splitting (5.18) holds if f_0 and m_s share the $2+N(N-1)/2$ conserved quantities, while (5.14) is more restrictive since the constraints are $6+N(N-1)/2$.

We now define the subset of $\mathcal{G}$ containing the Maxwellian solutions with the same total energy and angular momentum as f_0:

$$\mathcal{F} = \{m \in \mathcal{G} : E_{tot,m}(0) = E_{tot,f}(0), K_{jh,m}(0) = K_{jh,f}(0), \forall j, h\}. \tag{5.21}$$

In particular, we get $m_\infty, m_s \in \mathcal{F}$ and the following properties.

Corollary 2 *Under the setting of Lemma 7, we get*:

(a) *The inequality* (5.19) *implies*

$$H[m_s] \le H[\tilde{m}(t)], \quad \forall \tilde{m} \in \mathcal{F}, \tag{5.22}$$

$$H[m_s] = \inf\{H[m(t)], \forall m \in \mathcal{F}\}. \tag{5.23}$$

(b) *The relation* (5.13) *holds, since for each* $a, b \in \mathbb{R}$ *such that*

$$H[m_s] < a < H[m_\infty(t)], \quad H[m_\infty(t)] < b < H[f_0] \tag{5.24}$$

there exist m_a, m_b *in* $\mathcal{G}$ *with* $a = H[m_a(t)]$ *and* $b = H[m_b(t)]$.

(c) *If* $\tilde{m} \ne m_s$ *and* $\tilde{m} \in \mathcal{F}$, *then* $H[f(t), \tilde{m}(t)] \ne H[f(t)] - H[\tilde{m}(t)]$.

Proof (a) The inequality (5.22) is obvious since $f = \tilde{m}(t)$ is a particular solution of (5.1). Equation (5.23) follows from the continuous dependence of $H[m(t)]$ from its parameters $c_0, \dots, c_5$ (see (5.10) for $m_1 = m$). In the Appendix, we made explicit computation for the family $\mathcal{F}$. The entropy of $\tilde{m} \in \mathcal{F}_1$ is $H[\tilde{m}](0) = N(-1 + \frac{1}{2}\log(\frac{-c_0}{2\pi^2}))$ (cf. (7.2)), which is a continuous strictly decreasing function in c_0 (with $c_0 \le -1/2$ and $c_0 = -1/2$ only for m_s). Its minimum value is $H[m_s] = -N(1+\log(2\pi))$.

(b) From point (a) we can choose m_a, m_b in $\mathcal{F}$. In fact, for each $h \in \mathbb{R}$ such that $H[m_s] < h$ there exists m_h in $\mathcal{F}$ satisfying the equality $h = H[m_h(t)]$.

(c) By contradiction, if $H[f(t), \tilde{m}(t)] = H[f(t)] - H[\tilde{m}(t)]$ for some $\tilde{m}$, then the inequality (5.22) could be reversed by exchanging the role of m_s and $\tilde{m}(t)$, giving as result $H[\tilde{m}(t)] - H[m_s] = H[\tilde{m}(t), m_s] = 0$, which would imply $\tilde{m} = m_s$. □

Equation (5.23) shows that m_s is the state having the lowest entropy in $\mathcal{F}$.

For this reason one could be erroneously led to think that m_s is the most probable equilibrium state for (5.1), while we have shown that m_∞ instead is the right candidate for the relative entropy convergence, and also for the L^1-convergence (see Sect. 6).

As implicitly said in Lemma 7, the choice of f_0 determines whether or not m_∞ and m_s coincide.

In the following, we summarize all the information about the case $m_\infty = m_s$.

Corollary 3 *Under the setting of Lemma 7, let f_0 fulfil the additional assumptions*:

$$\begin{aligned} &J_{x,f}(0) = 0, \qquad J_{v,f}(0) = 0, \qquad L_f(0) = 0, \\ &E_{pot,f}(0) = E_{kin,f}(0) = \frac{E_{tot,f}(0)}{2}. \end{aligned} \tag{5.25}$$

Then,

(a) *The previous moments are conserved for each time* $t > 0$:

$$J_{x,f}(t) = 0, \qquad J_{v,f}(t) = 0, \qquad L_f(t) = 0,$$
$$E_{pot,f}(t) = E_{kin,f}(t) = E_{tot,f}(0)/2,$$

i.e. the moments do not oscillate during the evolution. In particular, we get equipartition between kinetic and potential energy for each time.

(b) $m_\infty(t,x,v) = m_s(x,v)$, *which implies* $H[m_\infty] = H[m_s]$, $H[m_s] < H[m], \forall m \in \mathcal{F} - \{m_s\}$, $H[f(t), m_s] < H[f(t), m], \forall m \in \mathcal{G} - \{m_s\}$.

Proof (a) follows from Lemma 4 and (b) is a consequence of Lemma 7 and the definition of m_∞. □

We conclude the section with an alternative proof of Proposition 1(c), which employs the elements of the family $\mathcal{F}$ as 'test functions'. This result is interesting since it is independent of the knowledge of Sects. 5.3 and 5.4. The computations used could turn useful in other cases. We show the case $m_* = m_s$. Hence (5.17) becomes

$$H[f_0, m_s] < H[f_0, m(0)], \quad \forall m \in \mathcal{G} - \{m_s\}. \tag{5.26}$$

Without restriction of generality we consider the following normalizations

$$D_f(0) = 1 \qquad E_{tot,f}(0) = N \qquad K_{jh,f}(0) = 0 \quad \forall j, h = 1, \ldots, N, \tag{5.27}$$

The same relation holds for $f(t)$ and for the stationary Maxwellian m_s, equal to $m_s = (2\pi)^{-N} \exp(-(|x|^2 + |v|^2)/2)$.

Proposition 2 *Let* $f_0 = f_0(x,v) \geq 0$ *be the initial datum of* (5.1) *with* $\Phi(x) = |x|^2/2$, *which satisfies* (5.27) *and let* $\rho(t,x), u(t,x)$ *be the density and the mean velocity of a corresponding solution* $f(t,x,v)$. *Then* (5.26) *is equivalent to*

$$2E_{pot,f}(0) = N, \qquad L_f(0) = 0, \qquad J_{v,f}(0) = 0, \qquad J_{x,f}(0) = 0. \tag{5.28}$$

Moreover, these conditions hold for all times: $\forall t \geq 0$

$$2E_{pot,f}(t) = N, \qquad L_f(t) = 0, \qquad J_{v,f}(t) = 0, \qquad J_{x,f}(t) = 0. \tag{5.29}$$

Proof

Case $t = 0$.

Step 1: (5.26) *implies* (5.28). We rewrite (5.26) as: $\exists f_0$ such that

$$\forall m \in \mathcal{G} - \{m_s\} \quad \int f_0 \log m_s dx dv > \int f_0 \log m(0) dx dv. \tag{5.30}$$

By (5.27) and since $m_s \in \mathcal{F}$, the l.h.s. is equal to $H[m_s]$. For the r.h.s. we obtain, using (5.8),

$$\int f_0 \log m(0) dx dv = -2c_1 \int \rho_0 |x|^2 dx + c_5 + 2N(c_0 + c_1) - c_4 \int \rho_0 (\mathbb{I} \cdot x) dx$$

$$+\int \rho_0(-2c_2x + c_3\mathbb{I})\cdot u_0\,dx + \sum_{j,h=1}^{N}\int \rho_0\, u_{0,j} w_{jh} x_h dx.$$

Equation (5.27) implies $\sum_{j,h=1}^{N} w_{jh}\int \rho_0\, u_{0,j}x_h dx = 0$. Then (5.30) can be rewritten as

$$-N(1+\log(2\pi)) - c_5 - 2Nc_0 > 2Nc_1 - 2c_1\int \rho_0|x|^2dx - c_4\int \rho_0(\mathbb{I}\cdot x)dx$$
$$+\int \rho_0(-2c_2x + c_3\mathbb{I})\cdot u_0dx. \tag{5.31}$$

We test first this condition in the subfamily $\mathcal{F}_1 - \{m_s\}$ of $\mathcal{G}$ (see Appendix 7.1) where we get:

$$2c_1\int \rho_0|x|^2dx > N(1+\log(2\pi)) + 2N(c_0+c_1) + \frac{N}{2}\log\left(\frac{-c_0}{2\pi^2}\right)$$

and two possible choices for c_1: $c_1 = -\sqrt{c_0(c_0+\frac{1}{2})} < 0$ and the positive one $\tilde{c}_1 = \sqrt{c_0(c_0+\frac{1}{2})}$. Then, we obtain the following constraint: $\forall c_0 < -1/2$,

$$N + \frac{N}{2\tilde{c}_1}\left(1 + 2c_0 + \frac{1}{2}\log(-2c_0)\right) < \int \rho_0|x|^2dx$$
$$< N + \frac{N}{2c_1}\left(1 + 2c_0 + \frac{1}{2}\log(-2c_0)\right).$$

We notice that the first term of the inequality is continuously increasing in c_0 while the last one is continuously decreasing in c_0. Therefore, if we take c_0 tending to $-1/2$ we obtain $\int \rho_0|x|^2dx = N$.

In the same way we obtain (see Appendix 7.1 for definitions): In $\mathcal{F}_a - \{m_s\}$:

$$\frac{2N + 4Nc_0 + N\log(-2c_0)}{[4c_0(2c_0+1)]^{1/2}} < \int \rho_0(\mathbb{I}\cdot x)dx < \frac{-2N - 4Nc_0 - N\log(-2c_0)}{[4c_0(2c_0+1)]^{1/2}}$$

and then $\int \rho_0(\mathbb{I}\cdot x)dx = 0$. In $\mathcal{F}_b - \{m_s\}$:

$$\frac{2N + 4Nc_0 + N\log(-2c_0)}{[4c_0(2c_0+1)]^{1/2}} < \int \rho_0(\mathbb{I}\cdot u_0)dx < \frac{-2N - 4Nc_0 - N\log(-2c_0)}{[4c_0(2c_0+1)]^{1/2}}$$

and then $\int \rho_0(\mathbb{I}\cdot u_0)dx = 0$. In $\mathcal{F}_c - \{m_s\}$:

$$\frac{2N + 4Nc_0 + N\log(-2c_0)}{2[2c_0(2c_0+1)]^{1/2}} < \int \rho_0(x\cdot u_0)dx < \frac{-2N - 4Nc_0 - N\log(-2c_0)}{2[2c_0(2c_0+1)]^{1/2}}$$

and then $\int \rho_0(x\cdot u_0)dx = 0$.

In all the 4 families we used the fact that $c_0 = -1/2 \Leftrightarrow m = m_s$.

Step 2: Equation (5.28) *implies* (5.26). Using (5.28) in (5.31) and considering the equivalence

$$\int f_0 \log m(0)dxdv = \int m_s \log m(0)dxdv = c_5 + 2Nc_0,$$

we reformulate (5.31) (equivalently (5.26)) as

$$H[m_s, m(0)] > 0 \quad \forall \mathcal{G} - \{m_s\}$$

which is always satisfied.

Case $t > 0$. The result can be shown in the same way, using Lemma 3a(ii) (with strict inequality) and the conservation laws. □

Remark 3 (a) Choice of f_0. The assumption (5.28) is obviously satisfied by $(\rho_0, u_0, T_0) = (\rho_s, 0, 1)$ (with $\rho_s(x) = \int_{\mathbb{R}^N} m_s dv$) and the only Maxwellian in $\mathcal{F}$ with these moments is m_s. A more general choice for the initial datum f_0 is given by an L^1-function symmetric in each of the space and velocity variables, with unit mass and potential energy equal to $N/2$.

(b) Radial solutions. Since radial solutions for (5.1) of type $f(t, x, |v|)$ satisfy (5.29) for every time, a natural guess could be to expect their convergence to m_s for $t \to \infty$. As proved in Lemma 8, the velocity moments of such radial solutions are stationary, i.e. $(\rho_0, 0, T_0) = (\rho(t), 0, T(t))$. Therefore $f(t, x, |v|)$ can tend to m_s only if the initial datum f_0 satisfies $(\rho_0, 0, T_0) = (\rho_s, 0, 1)$. Nothing is known about the existence of such solution. However, if a time-dependent radial solution existed in the case of the BGK equation, then we could prove convergence to equilibrium with exponential rate by means of the entropy. In fact, the hypothesis on the moments would imply that the local Maxwellian $M[f](t)$ coincides with m_s, and then (using (3.5)–(3.6))

$$\frac{d}{dt} H[f(t), m_s] \le -H[f(t), M[f](t)] = -H[f(t), m_s]$$

and $H[f(t), m_s] \le H[f(0), m_s]\, e^{-t}$.

(c) Anisotropic potential. The analysis for the anisotropic harmonic potential $\Phi(x) = \frac{1}{2}\sum_{k=1}^{N} a_k {x_k}^2$ (with $a_k > 0$ and at least two different coefficients $a_r \ne a_s$) leads to another family $\mathcal{F}(\Phi)$ of time dependent steady states with

$$c(t,x) = c_0, \qquad b_j(t,x) = c_3 \cos(\sqrt{a_j} t) + c_4 \sin(\sqrt{a_j} t) + \sum_{h=1}^{N} w_{jh} x_h,$$

$$a(t,x) = c_0 \sum_{k=1}^{N} a_k {x_k}^2 + \sum_{k=1}^{N} \big(\sqrt{a_k} x_k (c_3 \sin(\sqrt{a_k} t) - c_4 \cos(\sqrt{a_k} t))\big) + c_5$$

and $w_{jh} = 0$ if $a_h \ne a_j$. The hydrodynamical quantities are directly derived as in the isotropic case, showing that the temperature is constant. The computations of this section and of the Appendix can be performed even in this case, with small modifications. Mass, energy and entropy are: $M = (-\pi/c_0)^N (\prod_{k=1}^{N} \sqrt{a_k})^{-1} \exp(c_5 - (N c_3^2)/(2c_0) - (c_4^2)/(4c_0^2)) = 1$, $E_{tot} = N(c_3^2 - 4c_0 + 2c_0 c_4^2)/(8c_0^2) = N$, $H[\tilde{f}(0)] = 2c_0 N + c_5 - \frac{N}{2c_0}(c_3^2 + c_4^2)$. In particular, in the case of a totally anisotropic potential, the analogous of condition (5.28) becomes: $\int_{\mathbb{R}^N} \rho_0 u_0 \cdot \mathbb{I} dx = 0$, $\int_{\mathbb{R}^N} \rho_0 (\sum_k \sqrt{a_k} x_k) dx = 0$. The anisotropic harmonic trap has been investigated in [19] via an ansatz on the solution and numerical simulations.

6 BGK with Quadratic Potential

In this section we return to the study of the BGK equation in case of an harmonic potential $\Phi(x) = |x|^2/2$. Our aim is to understand if the necessary conditions (5.11) (both for the L^1

and the Lyapunov convergence of f to m_∞) can be sufficient. By Theorem 4 we know that f converges to a Maxwellian $m \in C([0,+\infty); L^1(\mathbb{R}^{2N}))$, solution of the equation

$$\partial_t m + v \cdot \nabla_x m - x \cdot \nabla_v m = 0,$$

and with $m \in \mathcal{G}$.

The question is now the identification of m.

By the moments estimates we surely know that, using the notations of Definition 1, for $t_n \to +\infty$ (up to subsequences)

$$J_{x,f}(t+t_n) \to J_{x,m}(t) \quad \text{and} \quad J_{v,f}(t+t_n) \to J_{v,m}(t) \quad a.e. \tag{6.1}$$

Then we note that the limit Maxwellian m of Theorem 4 has not necessarily the same energy and the same angular momentum as the initial datum f_0. This is due to a lack of control of the tails of the sequence $(|v|^2 + 2\Phi(x))f(t+t_n)$ in unbounded domains. Even the higher-order estimates in Theorem 2 cannot at present be useful since they are exponentially time-dependent. However, the pointwise convergence and the local weak convergence hold true. Furthermore the passage to the limit for the energy and the angular momentum is what one expects from the conservation laws.

Therefore we introduce the following a-priori assumptions on f: for a.e. $t \geq 0$, as $t_n \to +\infty$

$$\begin{aligned} L_f(t+t_n) &\to L_m(t), \\ E_{pot,f}(t+t_n) &\to E_{pot,m}(t), \\ E_{kin,f}(t+t_n) &\to E_{kin,m}(t), \\ K_{jh,f}(t+t_n) &\to K_{jh,m}. \end{aligned} \tag{6.2}$$

Anyway we underline that the identification of m_∞ and in general all results of Sect. 5 do not use (6.2) as assumption. The fact that the optimal Maxwellian m_∞ has the same energy and angular momentum as f_0 comes from the computations and it is not assumed.

Note that f is derived from an L^1-existence theory and the Maxwellians in $\mathcal{G}$ are classical solutions, therefore we implicitly assume the matching of these two theories. However, in Sect. 5 the continuity property of the Maxwellians has never been used.

We recall that m_∞ and m_s have been defined in the Sects. 5.3 and 5.5.

Theorem 5 *Assume $\Phi = |x|^2/2$. Under the setting of Theorem 4, we assume the conditions (6.2) for the BGK-solution f. Then,*

(a) *If f_0 satisfies the conditions of non oscillation (5.25), then for $t_n \to +\infty$*

$$f_n(t) \to m_s, \quad in\ L^1(\mathbb{R}^{2N})$$

where m_s is the stationary Maxwellian equilibrium.

(b) *(General case) If f_0 does not fulfil the conditions (5.25) and if we consider the sequence $t_n = 2\pi n$, then for $t_n \to +\infty$*

$$f_n(t) \to m_\infty(t), \quad in\ L^1(\mathbb{R}^{2N})$$

where $m_\infty(t)$ is the time periodic Maxwellian equilibrium.

Proof We recall that each $m \in \mathcal{G}$ depends on $6+N(N-1)/2$ parameters (including the one fixed by the convergence of the mass).

In case (a) the moments of f in Definition 1 are all constant in time. Due to (6.2) they are supposed to pass to limit as $t_n \to +\infty$. Hence, we have $6+N(N-1)/2$ constants which fix $m=m_\infty$ of Lemma 5 uniquely. And by Corollary 3 we know that this Maxwellian coincides with m_s.

In case (b) the moments of f are time dependent. Since for a general sequence t_n the limit m is defined up to a time shift, we cannot be sure "of the time of m". Without the time synchronization between f and m it would be very difficult to identify m. The choice $t_n = 2\pi n$ is made to prevent this occurrence. For such t_n, in (6.2) we get $L_f(t+t_n) = L_f(t)$, $E_{pot,f}(t+t_n) = E_{pot,f}(t)$, and so on. The moments remain time dependent, but for each time they result t_n-independent. Then we consider $t=0$ and we look for the limit $f(t_n) \to m(0)$. We get $L_f(t_n) = L_f(0) \to L_m(0) = L_f(0)$, and the same holds for the other moments in (6.2). Hence, f_0 and $m(0)$ fulfil (5.11) and $m=m_\infty$ is therefore uniquely defined (see also (5.9)). □

Remark 4 (a) Note that (6.2) is satisfied if, for some $\epsilon > 1$, one proves

$$\int_{t_1}^{t_2}\int_{\mathbb{R}^{2N}} f(t,x,v)(|v|^2+|x|^2)^\epsilon dt dx dv < g(t_2-t_1),$$

with $g(s)$ a continuous function in $\mathbb{R}$. In this way, the bound depends on the time difference (t_2-t_1) and therefore is uniform on compact time intervals. This is the same idea applied in Theorem 4 to get the unconditional convergence to m. Note that the (6.2) is anyway weaker than this condition.

(b) Another open question concerns the entropy convergence. From (5.14) we know that, as $t \to +\infty$, $H[f(t), m_\infty(t)] \to 0 \Leftrightarrow H[f(t)] \to H[m_\infty(t)]$ (in particular it holds for $m_\infty(t) = m_s$). For $t_n = 2\pi n$ we can also rewrite the expression as: $H[f(t+t_n), m_\infty(t)] \to 0 \Leftrightarrow H[f(t+t_n)] \to H[m_\infty(t)]$ for $t_n \to +\infty$, in order to employ the result of Theorem 4. Anyway, the open question consists only in proving the passage to the limit, since the identification of the Maxwellian equilibrium has already been solved in Theorem 5.

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Appendix

7.1 Computations for the Harmonic Potential

We collect here the computations used in Proposition 2 and partially in Corollary 2.

For convenience of the reader, we recall the following integration formulas. Let $x, \xi, \eta \in \mathbb{R}^N$, $\alpha \in (0,+\infty)$, then

$$\int_{\mathbb{R}^N} \exp(-\alpha|x|^2 + \xi\cdot x)|x|^2 dx = \frac{1}{2\alpha}\left(N + \frac{|\xi|^2}{2\alpha}\right)\left(\frac{\pi}{\alpha}\right)^{N/2} \exp\left(\frac{|\xi|^2}{4\alpha}\right),$$

$$\int_{\mathbb{R}^N} \exp(-\alpha|x|^2 + \xi \cdot x)(\eta \cdot x)dx = (\eta \cdot \xi)\frac{1}{2\alpha}\left(\frac{\pi}{\alpha}\right)^{N/2} \exp\left(\frac{|\xi|^2}{4\alpha}\right),$$

$$\int_{\mathbb{R}^N} \exp(-\alpha|x|^2 + \xi \cdot x)dx = \left(\frac{\pi}{\alpha}\right)^{N/2} \exp\left(\frac{|\xi|^2}{4\alpha}\right).$$

Next we consider the family $\mathcal{F}$, introduced in (5.21) satisfying the same normalizations (5.27) as f_0. Even if $K_{ji} = 0$ does not necessary imply $w = \{w_{ij}\} = 0$ in all $\mathcal{F}$, for the proof of Proposition 2 it is sufficient to consider only the subfamily with $w = 0$. In addition to the conservation laws, the elements of this subfamily fulfil some obvious requirements:

(i) $\tilde{T}(t) > 0$ implies $c(t) < 0$, $\forall t > 0$ and for $t = 0$ gives $c_0 \pm c_1 < 0 \Rightarrow c_0 < 0$;
(ii) $\tilde{\rho}(t, x) \in L^1(\mathbb{R}^N)$ implies $c_0^2 > c_2^2 + c_1^2 \to |c_1| < (c_0^2 - c_2^2)^{1/2}$;
(iii) $H[m_s, \tilde{m}(0)] \geq 0$ implies $c_5 + 2Nc_0 \leq -N - N\log(2\pi)$;
(iv) $H[\tilde{m}(0)] \geq H[m_s]$ is equivalent to $c_0^2 - c_2^2 - c_1^2 \geq \frac{1}{4}$, which gives $c_0 \leq -\frac{1}{2}$.

This last constraint is essential in the proof of Proposition 2 since it means $c_0 = -1/2 \Leftrightarrow \tilde{m} = m_s$ for the considered subfamilies. We now compute mass, entropy and energy assuming the normalizations (5.27) for $\tilde{m}$.

Mass in $\mathcal{F}$ with $w = 0$:

$$\int_{\mathbb{R}^N} \tilde{\rho}(0, x)dx = (2\pi\tilde{T}(0))^{N/2}\left(\frac{\pi}{\alpha}\right)^{N/2} \exp\left(\gamma + \frac{N\beta^2}{4\alpha}\right) = 1, \tag{7.1}$$

where $-\alpha = \frac{-c_2^2+c_0^2-c_1^2}{c_0+c_1}$, $\beta = -c_4 + \frac{c_2c_3}{c_0+c_1}$, $\gamma = c_5 - \frac{Nc_3^2}{4(c_0+c_1)}$, $\tilde{T}(0) = -\frac{1}{2(c_0+c_1)}$. The relation (7.1) can be used to express the coefficient c_5 in γ or c_4 in β with respect to the other ones.

Energy in $\mathcal{F}$ with $w = 0$:

$$\frac{1}{2}\int \tilde{\rho}(0, x)\big(|x|^2 + |\tilde{u}(0, x)|^2 + N\tilde{T}(0)\big)dx$$
$$= \frac{N}{2}\left[\frac{\delta}{2\alpha}\left(1 + \frac{\beta^2}{2\alpha}\right) - \frac{\eta\beta}{2\alpha} + \mu + \tilde{T}(0)\right] = N,$$

where $\delta = 1 + \frac{c_2^2}{(c_0+c_1)^2}$, $\eta = \frac{c_2c_3}{(c_0+c_1)^2}$, $\mu = \frac{c_3^2}{4(c_0+c_1)^2}$.

Entropy in $\mathcal{F}$ with $w = 0$:

$$H[\tilde{m}(0)] = -N - \frac{N}{2}\log\left(\frac{2\pi^2\tilde{T}(0)}{\alpha}\right). \tag{7.2}$$

Subfamilies of $\mathcal{F}$ Thanks to the previous relations for the mass and the energy, we find the expression for the 1-parameter subfamilies used in Sects. 5.2 and 5.4.

$$\mathcal{F}_1 = \{\tilde{m} \in \mathcal{F} | c_2 = c_3 = c_4 = 0, w = 0\} \quad \text{with } c_1^2 = c_0\left(\frac{1}{2} + c_0\right),$$

$$(\alpha, \beta, \delta, \eta, \mu, \gamma) = \left(c_1 - c_0, 0, 1, 0, 0, \frac{N}{2}\log\left(\frac{-c_0}{2\pi^2}\right)\right);$$

$$\mathcal{F}_a = \{\tilde{m} \in \mathcal{F} | c_1 = c_2 = c_3 = 0, w = 0\} \quad \text{with } {c_4}^2 = 4c_0\,(1 + 2c_0)\,,$$

$$(\alpha,\beta,\delta,\eta,\mu,\gamma)=\left(-c_0,-c_4,1,0,0,N\left(\log\left(\frac{-c_0}{\pi}\right)+\frac{{c_4}^2}{4c_0}\right)\right);$$

$$\mathcal{F}_b=\{\tilde{m}\in\mathcal{F}|c_1=c_2=c_4=0,\,w=0\}\quad\text{with } c_3^2=4c_0\,(1+2c_0)\,,$$

$$(\alpha,\beta,\delta,\eta,\mu,\gamma)=\left(-c_0,0,1,0,\frac{c_3^2}{4c_0^2},-N\log\left(\frac{\pi}{-c_0}\right)\right);$$

$$\mathcal{F}_c=\{\tilde{m}\in\mathcal{F}|c_1=c_3=c_4=0,\,w=0\}\quad\text{with } c_2^2=c_0\left(\frac{1}{2}+c_0\right),$$

$$(\alpha,\beta,\delta,\eta,\mu,\gamma)=\left(\frac{-c_0^2+c_2^2}{c_0},0,1+\frac{c_2^2}{c_0^2},0,0,-\frac{N}{2}\log\left(\frac{\pi^2}{c_0^2-c_2^2}\right)\right).$$

Radial solutions in velocity We conclude with the analysis of radial solutions, showing that they can converge to m_s together with their moments only if their initial hydrodynamical quantities coincide with those of m_s. In this case an exponential rate of convergence can be found for the BGK model (see Remark 3).

Lemma 8 *Let* $h_0=h_0(x,|v|)\geq 0, h_0\neq m_s$, *an initial condition such that* $\int_{\mathbb{R}^{2N}}h_0(X(-t),V(-t))dv\geq g(x)$ *with* $g>0, g\in L^1((1+\Phi(x))dx)$.

If (5.1) *with* $\Phi(x)=|x|^2/2$ *admits a radial solution in the* v *variable, namely* $f(t,x,v)=h(t,x,|v|)\in C([0,+\infty),L^1(\mathbb{R}^{2N}))$, *then its hydrodynamical quantities are constant in time* $(\rho(t),u(t),T(t))=(\rho_0,0,T_0)$.

Proof From hypothesis and the definition of the 1st moment $u(t,x)$, we obtain $\rho(t,x)\cdot u_k(t,x)=\int h(t,x,|v|)v_k dv=0, a.e.x, \forall t,\ 1\leq k\leq N$. Thus, for each t_1 fixed, either $\rho(t_1,x)=0$ or $u_k(t_1,x)=0$ can happen.

The hypothesis on h_0 and the mild formulation (2.4) implies $\rho(t,x)>0$ a.e. in x, for all times. With $u=0$ the hydrodynamical system becomes

$$\partial_t\rho+\nabla_x\cdot(\rho u)=0\quad\Rightarrow\quad\partial_t\rho=0$$
$$\frac{d}{dt}(\rho(x)T(t,x)N)=\int|v|^2(C(h)-v\cdot\nabla_x h+\nabla_x\Phi\cdot\nabla_v h)dv=0.$$

This means that h has moments $(\rho,u,T)=(\rho_0,0,T_0)$ independent of time. □

7.2 Approximate Solutions and Proof of Theorem 1

We construct the approximated solutions for the BGK Boltzmann equation (1.1)–(1.7) in analogy to [25]. One proceeds in two steps: first, one proves existence and uniform estimates for the model (BGK_ϵ^α), then one passes to the limit $\alpha\to 0$ and obtains a solution for the approximated model (BGK_ϵ). This model differs from (1.1) since a Dirac mass in v is avoided in the nonlinearity because $\gamma_\epsilon\geq\epsilon$. In the proof of Theorem 1 there is the last step of the procedure: the passage to the limit $\epsilon\to 0$ in the equation (BGK_ϵ), which gives the solution of the original problem (1.1)–(1.7).

Lemma 9 *For $\epsilon, \alpha \in (0,1]$ consider the model*

$$(BGK_\epsilon^\alpha) \quad \begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\Phi\cdot\nabla_v f = h_\alpha M_\epsilon^\alpha[f] - f, \\ M_\epsilon^\alpha[f](t,x,v) = \dfrac{\rho(t,x)}{(2\pi\gamma_\alpha(t,x))^{N/2}}\exp\Big(-\dfrac{|v-w_\alpha(t,x)|^2}{2\gamma_\alpha(t,x)}\Big), \\ h_\alpha(x) = \exp(-\alpha\Phi(x)), \\ w_\alpha(t,x) = u\inf(|u|, 1/\alpha)/|u|, \gamma_\alpha(t,x) = \inf(\sup(\epsilon, T), 1/\alpha), \\ \begin{pmatrix} \rho \\ \rho u \\ \rho|u|^2+\rho T \end{pmatrix}(t,x) = \int_{\mathbb{R}^N}\begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,x,v)dv \end{cases}$$

with initial condition (1.6)–(1.7) *and Φ satisfying* (1.3)–(1.5)*. Then, there exists a unique nonnegative mild solution f, with $(1+2\Phi(x)+|v|^2)f \in C([0,+\infty); L^1(\mathbb{R}^{2N}))$, such that, for all $t \le t_0$,*

$$\int_{\mathbb{R}^{2N}} \big(1+\Phi(x)+|v|^2+|\log f|\big) f(t,x,v)dxdv \le c(\epsilon, t_0),$$

$$\int_{\mathbb{R}^{2N}} \big(1+\Phi(x)+|v|^2+|\log(h_\alpha M_\epsilon^\alpha[f])|\big) h_\alpha(x) M_\epsilon^\alpha[f](t,x,v)dxdv \le c(\epsilon, t_0),$$

$$\int_{\mathbb{R}^{2N}} \big(1+\Phi(x)+|v|^2+|\log(M_\epsilon^\alpha[f])|\big) M_\epsilon^\alpha[f](t,x,v)dxdv \le c(\epsilon, t_0).$$

Lemma 10 *For $\epsilon \in (0,1]$ consider the model*

$$(BGK_\epsilon) \quad \begin{cases} \partial_t f + v\cdot\nabla_x f - \nabla_x\Phi\cdot\nabla_v f = M_\epsilon[f] - f \\ M_\epsilon[f](t,x,v) = \dfrac{\rho(t,x)}{(2\pi\gamma_\epsilon(t,x))^{N/2}}\exp\Big(-\dfrac{|v-u(t,x)|^2}{2\gamma_\epsilon(t,x)}\Big), \\ \gamma_\epsilon(t,x) = \sup(\epsilon, T), \\ \begin{pmatrix} \rho \\ \rho u \\ \rho|u|^2+\rho T \end{pmatrix}(t,x) = \int_{\mathbb{R}^N}\begin{pmatrix} 1 \\ v \\ |v|^2 \end{pmatrix} f(t,x,v)dv \end{cases}$$

with initial condition (1.6)–(1.7) *and Φ satisfying* (1.3)–(1.5).

Then, there exists a nonnegative mild solution f, with $(1+2\Phi(x)+|v|^2)f \in C([0,+\infty); L^1(\mathbb{R}^{2N}))$, such that, for all $t \le t_0$,

$$\int_{\mathbb{R}^{2N}} (1+\Phi(x)+|v|^2+|\log f|) f(t,x,v)dxdv \le c(t_0), \tag{7.3}$$

$$\int_{\mathbb{R}^{2N}} (1+\Phi(x)+|v|^2+|\log(M_\epsilon[f])|) M_\epsilon[f](t,x,v)dxdv \le c(t_0). \tag{7.4}$$

For the proof of Lemma 10 see Theorem 4 of [25].

Proof of Lemma 9 The proof is a consequence of a fixed point argument in the space $Y_{t_0} = C([0,t_0]; Z)$, with $Z = L^1((1+|v|^2+2\Phi(x))dxdv)$. In [25] it has been proved that map $z \mapsto M_\epsilon^\alpha[z]$ is Lipschitz continuous on $L^1((1+|v|^2)dv)$. Next, we show that the map $z \mapsto h_\alpha M_\epsilon^\alpha[z]$ is uniformly Lipschitz continuous on Z. From the pointwise inequality $h_\alpha(x)(1+$

$|v|^2+2\Phi(x)) \le c_2(\alpha)(1+|v|^2)$, with $c_2(\alpha) > 0$ a constant dependent of α, we can conclude, $\forall f_1, f_2 \in Z$,

$$\int_{\mathbb{R}^{2N}} |M_\epsilon^\alpha[f_1] - M_\epsilon^\alpha[f_2]| h_\alpha(x)(1+|v|^2+2\Phi(x))dxdv$$
$$\le c_2(\alpha) \int_{\mathbb{R}^N} \|(M_\epsilon^\alpha[f_1] - M_\epsilon^\alpha[f_2])(1+|v|^2)\|_{L^1(\mathbb{R}_v^N)} dx \le c(\alpha,\epsilon)\|f_1 - f_2\|_Z.$$

Thus (BGK_ϵ^α) has a mild solution, which preserves the positivity. Using the relations $|w_\alpha|^2 - |u|^2 \le 0$, $\gamma_\alpha - T \le \epsilon$ and the Gronwall's lemma we get an estimate of the Z-norm of f

$$\|f(t,\cdot)\|_Z \le \|f_0\|_Z + tN\epsilon\|f_0\|_{L^1} \le c(t_0) \tag{7.5}$$

independent of α and ϵ and which let us extend to $[0,+\infty)$ the temporal domain of definition. Concerning the estimate of the entropy, as in [25] one easily gets $\int_{\mathbb{R}^{2N}} f|\log f|dxdv \le c(\epsilon,\alpha,t_0)$. Because of h_α on the r.h.s. of the equation, we must carefully check that the bound of $\int_{\mathbb{R}^{2N}} f \log f dvdx$ is uniform in α. To do it we need the following inequality

$$\int_{\mathbb{R}^N} \log(\epsilon)\rho dx \le \int_{\mathbb{R}^N} \log(\gamma_\alpha)\rho dx \le \int_{\mathbb{R}^N} \gamma_\alpha \rho dx \le c(\epsilon, t_0), \tag{7.6}$$

which follows from: $\epsilon = \inf(1/\alpha,\epsilon) \le \inf(1/\alpha, \sup(\epsilon,T)) = \gamma_\alpha \le \sup(\epsilon,T)$,

$$\int_{\mathbb{R}^N} \rho(\gamma_\alpha - T)dx \le \epsilon \int \rho dx \le c(\epsilon,t_0),$$
$$\int_{\mathbb{R}^N} \rho\gamma_\alpha dx \le \int_{\mathbb{R}^N} \rho T dx + c(\epsilon,t_0) \le \int_{\mathbb{R}^{2N}} |v|^2 f dvdx + c(\epsilon,t_0) \le c(\epsilon,t_0).$$

Now remember $(x-y)\log y \le (x-y)\log x$ and (7.6), and consider:

$$\begin{aligned}
&\partial_t \int_{\mathbb{R}^{2N}} f \log f dvdx \\
&= \int_{\mathbb{R}^{2N}} (1+\log f)(M_\epsilon^\alpha[f]h_\alpha - f)dvdx \\
&\le \int_{\mathbb{R}^{2N}} \log(M_\epsilon^\alpha[f]h_\alpha)(M_\epsilon^\alpha[f]h_\alpha - f)dvdx = \int_{\mathbb{R}^{2N}} \alpha\Phi(x)(1-h_\alpha)\rho dx \\
&\quad + \int_{\mathbb{R}^{2N}} \Big(\log\rho - \log(2\pi\gamma_\alpha)^{N/2} - \frac{|v-w_\alpha|^2}{2\gamma_\alpha}\Big)(M_\epsilon^\alpha[f]h_\alpha - f)dvdx \\
&\le \int_{\rho\le 1} \rho\log\rho(h_\alpha - 1)dx + \frac{N}{2}\int_{\mathbb{R}^N} \log(2\pi\gamma_\alpha)\rho dx - \frac{N}{2}\int_{\mathbb{R}^N} \log(\epsilon)\rho h_\alpha dx \\
&\quad + \int_{\mathbb{R}^{2N}} \frac{|v|^2+|u|^2}{\epsilon} f dvdx + c(\epsilon,t_0) \\
&\le \int_{e^{-\Phi(x)}\le\rho\le 1} \Phi(x)(1-h_\alpha)\rho dx + \int_{0\le\rho\le e^{-\Phi(x)}} \rho\log\frac{1}{\rho}(1-h_\alpha)dx + c(\epsilon,t_0)
\end{aligned}$$

which is less than $c(\epsilon,t_0)$. This concludes the estimate of the entropy of f. Similarly, one gets $\int_{\mathbb{R}^{2N}} |J[M_\epsilon^\alpha[f]h_\alpha]|dvdx \le c(\epsilon,t_0)$, for $J[f] = f\log f$. □

Proof of Theorem 1

Passing to the limit $\epsilon \to 0$. It is straightforward in analogy to Theorem 1 of [25] (see also [12]) and we skip the details. Calling $\{f_\epsilon\}_{\epsilon\in(0,1]}$ the sequence of solutions of the (BGK_ϵ) equation and $\{M_\epsilon[f_\epsilon]\}_{\epsilon\in(0,1]}$ the corresponding modified local Maxwellians, one proves that for $\epsilon \to 0$, due to (7.3), (7.4) and Step 1, it results

$$\begin{aligned}
& f_\epsilon \rightharpoonup f \quad \text{in weak} - L^1([0,t_0]\times\mathbb{R}^{2N}),\\
& \int_{\mathbb{R}^N}(1,v)f_\epsilon dv \to \int_{\mathbb{R}^N}(1,v)f dv \quad \text{in } L^1([0,t_0]\times\mathbb{R}^N),\\
& \int_{\mathbb{R}^N}|v|^2 f_\epsilon dv \to \int_{\mathbb{R}^N}|v|^2 f dv \quad \text{in } L^1([0,t_0]\times K_x),\\
& (\rho_\epsilon,\ \rho_\epsilon u_\epsilon) \to (\rho,\ \rho u) \quad \text{in } L^1([0,t_0]\times\mathbb{R}^N),\\
& \rho_\epsilon(|u_\epsilon|^2+NT_\epsilon) \to \rho(|u|^2+NT) \quad \text{in } L^1([0,t_0]\times K_x),\\
& M_\epsilon[f_\epsilon] \to M[f] \quad \text{in } L^1([0,t_0]\times\mathbb{R}^{2N}),
\end{aligned} \tag{7.7}$$

up to subsequences, where (u_ϵ, T_ϵ) and $(u,T) = (\frac{\int vf}{\rho}, \frac{\int|v-u|^2 f}{N\rho})$ are defined only where $\rho_\epsilon \neq 0$ and resp. $\rho \neq 0$. For $\rho = 0$ it results $M[f]=0$. Then, f is distributional and mild solution of (1.1).

Properties of the solutions. We write (1.1) in distributional form

$$\begin{aligned}
& -\int_0^\infty\int_{\mathbb{R}^{2N}}(\partial_t + v\cdot\nabla_x - \nabla_x\Phi(x)\cdot\nabla_v)\eta(t,x,v)f(t,x,v)dtdxdv\\
&\quad -\int_{\mathbb{R}^{2N}}\eta(0,x,v)f_0(x,v)dxdv = \int_0^\infty\int_{\mathbb{R}^{2N}}\eta(t,x,v)(M[f]-f)(t,x,v)dtdxdv
\end{aligned} \tag{7.8}$$

with $\eta(t,x,v) \in C_0^\infty([0,\infty)\times\mathbb{R}_x^N\times\mathbb{R}_v^N)$. Here and in the following we denote by β_R the cut-off function $\beta_R(z) = \chi(|z|/R)$, with $\chi \in C_0^\infty(\mathbb{R})$, $\chi \equiv 1$ for $|z| \le 1$ and $\chi \equiv 0$ for $|z| \ge 2$, $0 \le \chi \le 1$.

Bounds and temporal regularity: The bounds (2.1) come from the weak convergence. If $\int_{\mathbb{R}^{2N}}|x|^2 f_0 dxdv \le c_0$, then one easily gets $\int_{\mathbb{R}^{2N}}|x|^2 f(t)dxdv \le c(t_0)$ for $t\in[0,t_0]$. Concerning the temporal regularity, one can prove that $(1+|v|^2+2\Phi(x))f(t,x,v) \in C([0,\infty); L^1(\mathbb{R}^{2N}))$ by adapting Theorem 3.5.1 of [26].

Hydrodynamical system (2.2): The first and second equation in (2.2) are straightforward, while the third equation needs a bit of care. Here we choose $\eta(t,x,v) = (|v|^2 + 2\Phi(x))\alpha(t,x)\beta_R(v)$ as test function in (7.8), with $\alpha \in C_0^\infty([0,\infty)\times\mathbb{R}^N)$ and let $R\to+\infty$. With (2.1) we control each term of (7.8) except for $\int|v|^2 v\cdot\nabla_x\alpha(t,x)\beta_R(v)f dxdvdt$, which requires a bound of the third velocity moment $\int_{[0,t_0]\times K_x\times\mathbb{R}_v^N}|v|^3 f dxdvdt < c(t_0,K_x)$ on compact x-domains K_x. According to Lemma 1, the latter estimate holds for potentials with $\sigma = 1$ in (1.4). For other potentials the estimate holds under further hypotheses on f_0 (see Theorem 2).

Global conservation of mass and total energy: In case of the energy, we consider in (7.8) the test function $\eta(t,x,v) = \alpha(t)(|v|^2+2\Phi(x))\beta_R(|v|^2+2\Phi(x)) \in C_c^2([0,\infty)\times\mathbb{R}^{2N})$

supported on energy levels

$$\beta_R(|v|^2 + 2\Phi(x)) = 1 \quad \text{for } |v|^2 + 2\Phi(x) \leq R,$$
$$\beta_R(|v|^2 + 2\Phi(x)) = 0 \quad \text{for } |v|^2 + 2\Phi(x) \geq 2R. \tag{7.9}$$

Indeed the assumption (1.5) assures that Γ_R (for $R \geq R^*$) is an energy level submanifold with $C^2(\mathbb{R}^{2N-1})$ regularity, which is bounded in the phase-space since $\Phi(x) \to +\infty$ as $|x| \to +\infty$. With such a choice we have $(v \cdot \nabla_x - \nabla_x \Phi \cdot \nabla_v)\big((|v|^2 + 2\Phi(x))\beta_R(|v|^2 + 2\Phi(x))\big) = 0$. For $R \to +\infty$, $\beta_R(|v|^2 + 2\Phi(x))$ tends to 1 and the r.h.s. in (7.8) vanishes since $M[f]$ and f have the same velocity moments.

Global conservation of angular momentum for radial potential: Consider in (7.8) the test function $\eta(t, x, v) = (x_j v_k - x_k v_j)\alpha(t)\beta_R(v)\beta_R(x)$. Since Φ is radial, the two terms

$$|v \cdot \nabla_x \eta f| \leq |x||v|^2 \frac{c}{R} 1_{\{R \leq |x| \leq 2R\}} \alpha(t)\beta_R(v) f \leq c|v|^2 f,$$
$$|\nabla_x \Phi \cdot \nabla_v \eta f| \leq c|x||v||\nabla_x \Phi| \frac{c}{R} 1_{\{R \leq |v| \leq 2R\}} \alpha(t) f \leq c(1 + \Phi(x)) f$$

have an L^1-dominating function independent of R, where we used $\frac{|z|}{R} 1_{\{R \leq |z| \leq 2R\}} \leq 2$ and the first assumption in (1.4). Thus $\int_{[0,+\infty)\times\mathbb{R}^{2N}} (v \cdot \nabla_x \eta f - \nabla_x \Phi \cdot \nabla_v \eta) f(t, x, v) dx dv dt \to 0$ for $R \to +\infty$. The right hand side of (7.8) vanishes because $M[f]$ and f have the same velocity moments, and it is finite in case the potential has at least quadratic growth at infinity. Otherwise, we have to assume $\int_{\mathbb{R}^{2N}} |x|^2 f_0 dx dv \leq c_0$. □

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